Random regular graphs of non-constant degree: connectivity and Hamiltonicity

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Abstract

Let G_r denote a graph chosen uniformly at random from the set of r-regular graphs with vertex set $\{1, 2, \ldots, n\}$ where $3 \leq r \leq c_0 n$ for some small constant c_0. We prove that with probability tending to 1 as $n \to \infty$, G_r is r-connected and Hamiltonian.

1 Introduction

The properties of random r-regular graphs have received much attention. For a comprehensive discussion of this topic, see the recent survey by Wormald [22] or Chapter 9 of the book, *Random Graphs*, by Janson, Łuczak and Ruciński [12].

A major obstacle in the development of the subject has been a lack of suitable techniques for modelling simple random graphs over the entire range, $0 \leq r \leq n-1$, of possible values of r. The classical method for generating uniformly distributed simple r-regular graphs, is by rejection sampling using the configuration model of Bollobás [3]. The configuration model is a probabilistic interpretation of a counting formula of Bender and Canfield [2]. The method is most easily applied when r is constant or grows slowly with n, the number of vertices, as n tends to infinity. The formative paper [3] on this topic considered the case where $r = O((\log n)^{1/2})$. McKay [16] and McKay

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and Wormald [17, 18] subsequently gave alternative approaches which are useful for $r = o(n^{1/2})$ or $r = \Omega(n)$.

We use *edge switching techniques* extensively in this paper and note that these techniques have been successfully applied in a number of places e.g. [16], [17, 18], [9], [14] and [13].

Let G_r denote a graph chosen uniformly at random from the set $\mathcal{G}_r$ of simple r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. We consider properties of simple r-regular graphs for the case where $r \to \infty$ as $n \to \infty$, but $r = o(n)$. The properties we study are vertex r-connectivity and Hamiltonicity. These properties are also studied, in a recent paper by Krivelevich, Sudakov, Vu and Wormald [13], for the case where $r(n) \geq \sqrt{n} \log n$. Our paper complements [13] both in both in the range of r studied and in the techniques applied.

Theorem 1 *Assume $3 \leq r \leq c_0 n$ for some small positive absolute constant c_0. Then with probability tending to 1 as $n \to \infty$,*

(a) *G_r is r-connected.*

(b) *G_r is Hamiltonian.*

The results of Theorem 1 are well known for r constant. Result (a) is from Bollobás [4] and (b) is from Robinson and Wormald [20, 21], Bollobás [5], Fenner and Frieze [8]. For $r = o(n^{1/2})$ such results could have been proved with the help of the models of [16] and [17]. In fact this was done, for Hamiltonicity, up to $r = o(n^{1/5})$, in an unpublished work by Frieze [9], and for r-connectivity, up to $r \leq n^{.002}$ by Łuczak [15].

As [13] proves the case where $r \geq n^{1/2} \log n$, this implies G_r is r-connected and Hamiltonian **whp**[1] for all $3 \leq r \leq n - 4$.

2 Generating graphs with a fixed degree sequence.

Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, and let $2D = (d_1 + d_2 + \cdots + d_n)$. Let $\mathcal{G}_{\mathbf{d}}$ be the set of simple graphs G with vertex set $V = [n]$, degree sequence $\mathbf{d}$, and D edges.

Let Ω be the set of all $(2D)!/(D!2^D)$ partitions of $W = [2D]$ into D 2-element sets. An element of Ω is a *configuration*. The constituent 2-element sets of a configuration F are referred to as the *edges* of F.

Let $W_1, W_2, ..., W_n$ be the natural ordered partition $P_{\mathbf{d}}$ of $W = [2D]$ into sets of size $|W_i| = d_i$, and where $(\max W_i) + 1 = \min W_{i+1}$ for $i < n$.

[1] A sequence of events $\mathcal{E}_n$ is said to occur *with high probability* (**whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

Let $\Omega_{\mathbf{d}}$ be Ω with the understanding that the underlying set W is partitioned into P_d. The *degree sequence* of an element F of $\Omega_{\mathbf{d}}$ is $\mathbf{d}$. We often write Ω for $\Omega_{\mathbf{d}}$ when the context is clear. Define $\phi_{P_{\mathbf{d}}} : W \to [n]$ by $\phi(w) = i$ if $w \in W_i$. Let $\gamma(F)$ denote the multigraph with vertex set $[n]$ and edge multiset $E_F = \{\{\phi(x), \phi(y)\} : \{x, y\} \in F\}$.

Definition: Let $\Omega^*_{\mathbf{d}}$ denote those configurations F for which $\gamma(F)$ is simple relative to $P_{\mathbf{d}}$.

Remark 1 Note that each member of $\mathcal{G}_{\mathbf{d}}$ is the image under γ of precisely $\prod_{i=1}^n d_i!$ members of $\Omega^*_{\mathbf{d}}$. Thus sampling F uniformly from $\Omega^*_{\mathbf{d}}$ induces the uniform measure on $\gamma(F)$ and is equivalent to sampling uniformly from $\mathcal{G}_{\mathbf{d}}$.

If $d_i = r$, $(1 \le i \le n)$ we will say the configuration, F, is r-regular. The probability $|\Omega^*|/|\Omega|$ that the underlying r-regular multigraph $\gamma(F)$ of such a configuration F is simple is $\exp(-\Theta(r^2))$. For $r = o(n^{1/2})$ this follows from [17, 18] and for larger values of r from Lemma 2 below. This result allows us to prove many results directly via configurations and then condition the probability estimates for simple graphs.

Lemma 1 *Let* $\Delta = \max_{i\in[n]} d_i$. *Suppose that* $\Delta \le n/1000$ *and that* $\mathbf{d}$ *satisfies* $\min_{i\in[n]} d_i \ge \Delta/4$. *Given* $a, b \in [n]$, *if* G *is sampled u.a.r. from* $\mathcal{G}_{\mathbf{d}}$, *then*

$$\mathbf{Pr}(\{a, b\} \in E(G)) \le \frac{20\Delta}{n}.$$

Proof Let

$$\Omega_1 = \{G \in \mathcal{G}_{\mathbf{d}} : \ \{a, b\} \in E(G)\} \text{ and } \Omega_2 = \mathcal{G}_{\mathbf{d}} \setminus \Omega_1.$$

We consider the set X of pairs $(G_1, G_2) \in \Omega_1 \times \Omega_2$ such that G_2 is obtained from G_1 by deleting disjoint edges $\{a, b\}, \{x_1, y_1\}, \{x_2, y_2\}$ and replacing them by $\{a, x_1\}, \{y_1, y_2\}$, $\{b, x_2\}$. Given G_1, we can choose $\{x_1, y_1\}, \{x_2, y_2\}$ to be any ordered pair of disjoint edges which are not incident with a, b or their neighbours and such that $\{y_1, y_2\}$ is not an edge of G_1. Thus each $G_1 \in \Omega_1$ is in at least $(D - (2\Delta^2 + 1))(D - (4\Delta^2 + 2))$ pairs. Each $G_2 \in \Omega_2$ is in at most $2D\Delta^2$ pairs. The factor of 2 arises because a suitable edge $\{y_1, y_2\}$ of G_2 has an orientation relative to the switching back to G_1. As $D \ge n\Delta/8$ it follows that

$$\frac{|\Omega_1|}{|\Omega_2|} \le \frac{2D\Delta^2}{(D - (2\Delta^2 + 1))(D - (4\Delta^2 + 2))} \le \frac{20\Delta}{n}.$$

□

Lemma 2 *Suppose* $100 \le r \le n/1000$. *Let* $d_j = r$, $1 \le j \le n$. *If* F *is chosen uniformly at random (u.a.r) from* Ω *then for* n *sufficiently large,*

$$\mathbf{Pr}(F \in \Omega^*) \ge e^{-2r^2}.$$

Proof Consider the following algorithm from Frieze and Łuczak [11]:

Algorithm GENERATE
begin
$D := rn/2$
$F_0 := \emptyset$
Let $\boldsymbol{\sigma} = (x_1, x_2, \ldots, x_{2D-1}, x_{2D})$ be an ordering of W
For $i = 1$ **to** D **do**
begin

$$F_i := \begin{cases} F_{i-1} \cup \{\{x_{2i-1}, x_{2i}\}\} & \left(\text{With probability } \frac{1}{2i-1}\right) \ \mathbf{A} \\ F_{i-1} \cup \{\{x_{2i-1}, z_1\}, \{x_{2i}, z_2\}\} - \{z_1, z_2\} & \left(\text{With probability } \frac{2i-2}{2i-1}\right) \ \mathbf{B} \end{cases}$$

Here $\{z_1, z_2\}$ is chosen u.a.r from F_{i-1} and z_1 is chosen u.a.r from $\{z_1, z_2\}$.
end
Output $F := F_D$
end

We first prove that GENERATE produces a u.a.r member of Ω whatever the ordering $\boldsymbol{\sigma} = (x_1, x_2, \ldots, x_{2D})$ of W. We then describe an ordering $\boldsymbol{\sigma}$ from which we can prove the lemma.

Let $W^{(i)} = (x_1, x_2, \ldots, x_{2i})$ and let Ω_i be the set of configurations of $W^{(i)}$. We show inductively that F_i is a random member of Ω_i. This clearly true for $i = 1$ and so assume that for some $i \geq 2$ we have that F_{i-1} is chosen u.a.r from Ω_{i-1}.

Now consider a bipartite graph H with vertex bipartition (Ω_{i-1}, Ω_i) and an edge (F, F') whenever $F' = F \cup \{x_{2i-1}, x_{2i}\}$ or $F' = (F \setminus \{a, b\}) \cup \{\{a, x_{2i-1}\}, \{b, x_{2i}\}\}$ for some $\{a, b\} \in F$. Each $F \in \Omega_{i-1}$ has degree $2i - 1$ in H and each $F' \in \Omega_i$ has degree 1. Our algorithm chooses F uniformly from Ω_{i-1} (induction) and then uniformly chooses an H-edge leaving F. This implies uniformity in Ω_i.

Label the configuration points in set W_k of the partition, as $\{(k-1)r + j : 1 \leq j \leq r\}$. For the ordering $\boldsymbol{\sigma}$ of W, we specify that x_i is always chosen as one of the remaining points for which $\phi(x_i)$ occurs as little as possible in the sequence $(\phi(x_1), \ldots, \phi(x_{i-1}))$. To be specific, when $i = (j-1)n + k$, $(1 \leq k \leq n,\ 1 \leq j \leq r)$, define x_i to be the point in W_k with label $(k-1)r + j$.

Let $\Omega_i^* = \{F \in \Omega_i : \gamma(F) \text{ is simple}\}$. Let $\Delta_i = \lceil 2i/n \rceil$ denote the maximum degree in $\gamma(F_i)$. Let the edge $\{\phi(x_{2i-1}), \phi(x_{2i})\} = \{a, b\}$ and let $\{\phi(z_1), \phi(z_2)\} = \{c, d\}$. We will prove that

$$\mathbf{Pr}(F_i \in \Omega_i^* \mid F_{i-1} \in \Omega_{i-1}^*) \geq \begin{cases} 1 & 2i \leq n \\ \left(1 - \frac{60\Delta_i}{(2i-1)n} - \frac{2\Delta_i^2 + 2\Delta_i}{i-1}\right) & n < 2i \leq rn. \end{cases} \qquad (1)$$

If $i \le n/2$ then F_i induces a matching. If $i > n/2$ and if at the ith step of GENERATE, $\{a,b\}$ already exists in Case A or is equal to $\{c,d\}$ in Case B then F_i will not be simple. The probability the edge $\{a,b\}$ exists, in the corresponding simple random graph, is at most $\frac{20\Delta_i}{n}$, by Lemma 1. Thus the probability the edge exists (Case A) or exists and is selected (Case B) is at most

$$\frac{20\Delta_i}{n}\left(\frac{1}{2i-1}+\frac{2i-2}{2i-1}\frac{1}{i-1}\right)=\frac{60\Delta_i}{(2i-1)n}.$$

Assume now that the ith step is type B and $\{a,b\} \neq \{c,d\}$.

When $\{a,b\} \cap \{c,d\} \neq \emptyset$, a loop may be created. This happens with probability at most $2\Delta_i/(i-1)$.

When one of a, b is adjacent to c or d, a parallel edge may be created. This happens with probability at most $2\Delta_i^2/(i-1)$.

All cases have been covered and the result follows from iterating (1) for $i \le rn/2$. □

Remark 2 In Lemma 7 we need to run algorithm GENERATE starting with a configuration F_0 on $[2D']$ and and restricting our random choice of $\{z_1, z_2\}$ to $F \setminus F_0$. The output is then F_0 plus a random configuration on $W = [2D'+1, 2D'+2D]$.

At this point we describe a simpler algorithm CONSTRUCT for obtaining a u.a.r configuration.

Algorithm CONSTRUCT

begin

$F_0 := \emptyset$; $R_0 := W := [2D]$

For $i = 1$ **to** D **do**

begin

Choose $u_i \in R_{i-1}$ *arbitrarily*

Choose v_i *uniformly at random* from $R_{i-1} \setminus \{u_i\}$

$F_i := F_{i-1} \cup \{\{u_i, v_i\}\}$; $R_i := R_{i-1} \setminus \{u_i, v_i\}$

end

Output $F := F_D$.

end

Remark 3 Neither of the algorithms generating F_D use any information about the partition $P_{\mathbf{d}}$ associated with the configuration. After iteration i, F_i is a u.a.r element of Ω_i. We can, if we wish, complete a certain number I of iterations using CONSTRUCT and then switch to GENERATE. Instead of initializing the ordering $\boldsymbol{\sigma}$ used in algorithm GENERATE with W we initialize $\boldsymbol{\sigma}$ with R_I, the remaining unmatched points.

3 r-Connectivity

We now prove Theorem 1(a). Since the result is already known for r constant, we can assume that $10^6 \le r \le c_0 n$, where c_0 is sufficiently small.

For a simple graph G with edge set E, the disjoint neighbour set, $N(S)$, of a set of vertices S is defined as $N(S) = \{w \notin S : \ \exists v \in S \text{ s.t. } \{v, w\} \in E\}$. When S is a singleton $\{v\}$ we use the notation $N(v)$.

Lemma 3 *Let $\mathcal{Q}_1 \subseteq \mathcal{G}_r$ be the event that for all vertices $v, w \in V$ of G_r:*

(a) *If $r = o(n)$ then $|N(v) \cap N(w)| \le 10 + o(r)$.*

(b) *If $\log^2 n \le r \le n$ then $|N(v) \cap N(w)| \le r^2/n + 5\sqrt{r \log n}$.*

Then $\mathbf{Pr}(\overline{\mathcal{Q}_1}) = O(1/n^2)$.

Proof Throughout this proof, we fix a vertex v and the set $S = N(v)$, of vertices which are the (disjoint) neighbours of v. Let w be a fixed vertex of $V - v$.

Let $\mathcal{F}(S) = \{G : G = G_r - v, \ N(v) = S\}$ be the set of graphs G with vertex set $V - v$ formed by deleting v from those r-regular graphs, G_r, for which $N(v) = S$. Thus $|S| = r$, and the vertices in S have degree $r - 1$ in G.

The vertex w partitions $\mathcal{F}$ into sets $\mathcal{F}(k) = \{G : \ |N(w) \cap S| = k\}$ where $0 \le k \le r$ if $w \notin S$ and $0 \le k \le r - 1$ if $w \in S$.

For sets $R, T \subseteq V - v$ let $\mathcal{N}(R, T) = \mathcal{N}(R, T; \ S, w)$ be the set of graphs in $\mathcal{F}$ with $N(w) \cap S = R$ and $N(w) - S = T$. If $|R| < |S - w|$, choose $x \in (S - w) \setminus R$ and $a \in T$. We consider a bipartite graph $\mathcal{B}$ with left vertex set $\mathcal{N}(R, T)$ and right vertex set $\mathcal{N}(R + x, T - a)$.

If $G \in \mathcal{N}(R, T)$ and $\{w, a\}, \{x, b\}$ are edges of G we make a *switching* $G : (wa, xb) \to (wx, ab)$ in which edges $\{w, a\}, \{x, b\}$ are replaced by $\{w, x\}, \{a, b\}$ provided the resulting graph G' is simple. These switchings define the edges of $\mathcal{B}$, and $d_L(G)$ (resp. $d_R(G')$) is the number of edges incident with G (resp. G') in $\mathcal{B}$.

Let $\nu(a, x; G) = |N(a) \cap N(x)|$ be the number of common neighbours of a and x in G. Let $\delta(a, x; G) = 1$ if $a \in N(x)$.

Considering the possibilities for b when the switching $G : (wa, xb) \to (wx, ab)$ gives a simple G' we have

$$d_L(G) = |N(x)| - \nu(a, x; G) - \delta(a, x; G)$$

for G' is simple iff $b \neq a$ and $b \notin N(a)$. Here $|N(x)| = r - 1$ as $x \in S$. The switching leaves $\delta(a, x; G') = \delta(a, x; G)$ and $\nu(a, x; G') = \nu(a, x; G)$ as $(\{a\} \cup N(a)) \cap N(x)$ is the same set in both graphs.

Considering the switching $G' : (wx, ab) \to (wa, xb)$ giving G we have

$$d_R(G') = |N(a)| - \nu(a, x; G') - \delta(a, x; G').$$

We note that $|N(a)| = r$ as $a \notin S$.

The graph $\mathcal{B}$ consists of components within which δ, ν (and hence d_L, d_R) are invariant. Consider a component with bipartition size (N_L, N_R). We now prove that $N_L \geq N_R$. In any component with edges we have $d_R = d_L + 1$ so that $N_R = N_L d_L/(d_L + 1)$. The case $(N_L, N_R) = (0, 1)$ of isolated vertices in the right bipartition, cannot occur. For, in G',

$$\nu(a, x; G') + \delta(a, x; G') \leq |N(x) - w| = r - 2$$

and so

$$d_R(G') = |N(a)| - \nu - \delta \geq 2.$$

Thus

$$|\mathcal{N}(R, T)| \geq |\mathcal{N}(R + x, T - a)|.$$

Given S and w, the size of $\mathcal{N}(R, T;\ S, w)$ is invariant for all $R, T, |R| = k$ by a simple symmetry argument.

Let $|\mathcal{N}(R, T;\ S, w)| = \eta(k)$. Thus $\eta(k)$ is a non-increasing function of k. Let $f(k) = |\mathcal{F}(k)|$ be the number of graphs in $\mathcal{F}$ with $|N(w) \cap S| = k$. If $w \notin S$ then for all $k \geq 0$, $f(k) = \binom{r}{k}\binom{n-2-r}{r-k}\eta(k)$. Similarly, if $w \in S$ then for all $k \geq 0$, $f(k) = \binom{r-1}{k}\binom{n-1-r}{r-1-k}\eta(k)$.

Suppose G is chosen u.a.r. from $\mathcal{F}(S)$ and let $Z(G) = |R|$. Then $\mathbf{Pr}(Z = k) = f(k)/|\mathcal{F}|$. Writing $N = n - 2, \rho = r - 1_{w \in S}$,

$$\mathbf{Pr}(Z = k) = \binom{\rho}{k}\binom{N - \rho}{\rho - k}\frac{\eta(k)}{|\mathcal{F}|}.$$

Let X be a hypergeometric random variable with $\mathbf{Pr}(X = k) = \binom{\rho}{k}\binom{N-\rho}{\rho-k}/\binom{N}{\rho}$. Then $\mathbf{Pr}(Z = k)/\mathbf{Pr}(X = k)$ decreases with k. It follows that $\mathbf{Pr}(Z \geq k) \leq \mathbf{Pr}(X \geq k)$ for any k.

The hypergeometric random variable X has mean $\mu = \rho^2/N$. The proportional error in bounding $\mathbf{Pr}(X = j)$ above by $\mathbf{Pr}(Y = j)$, where Y is the binomial random variable $B(\rho, \rho/N)$, is at most $\exp(\rho^2/(N - \rho))$ (see [7] p57). Thus provided $r = o(\sqrt{n})$, using the following bound (2) on Binomial tails (see [1]),

$$\mathbf{Pr}(Y \geq \beta\mu) \leq \left(\frac{e}{\beta}\right)^{\beta\mu} \tag{2}$$

we see that

$$\mathbf{Pr}(X \geq \beta\mu) \leq 2\left(\frac{e}{\beta}\right)^{\beta\mu}.$$

If $r \leq \log^2 n$ let $k = \alpha\rho + 10,\ \alpha = 1/\log\log n$, then

$$\mathbf{Pr}(X \geq \alpha\rho + 10) \leq 2\left(\frac{e\rho^2}{(\alpha\rho+10)(n-2)}\right)^{\alpha\rho+10} = o(n^{-4}).$$

For $\log^2 n \leq r \leq n$ let $k = \rho^2/(n-2) + 4\sqrt{\rho\log n}$. We can apply Azuma's inequality to the $0,1$ sequence of observations of the sampling process of X, with $c_i = 1$ to infer that

$$\mathbf{Pr}(X \geq \rho^2/(n-2) + 4\sqrt{r\log n}) = o(n^{-4}).$$

Note that if $r \geq \log^2 n$ and $r = o(n)$ then the bound in (b) implies that in (a). $\square$

Lemma 4 *Let $\mathcal{Q}_2$ be the event that no set of vertices $U \subset V$ of G_r, $1 \leq |U| \leq n/70$, induces more than $r|U|/12$ edges. Then* $\mathbf{Pr}(\mathcal{Q}_2) = 1 - O(1/n^2)$.

Proof Let $\beta = 1/12$ and $\theta = 1/70$. Let $|U| = u$.

Note first that in a simple r-regular graph a set of size u induces at most $\binom{u}{2}$ edges and, provided $u \leq 2\beta r$,

$$\binom{u}{2} \leq \beta r u.$$

Let $\mathcal{E} = \{F \in \Omega^* :$ No vertex set $U,\ 2\beta r \leq |U| \leq \theta n$ induces more than $\beta r|U|$ edges $\}$. It suffices to prove that $\mathbf{Pr}(\overline{\mathcal{E}}) = O(n^{-2})$.

In Ω the number of edges X falling inside a set U is dominated by a binomial random variable $Y \sim B(ur, u/(n-u))$ in which each configuration point of U independently selects a pairing on the assumption that all configuration points of U are available, and that ru configuration points of $V \setminus U$ are unavailable. Now, $\mathbf{E}Y = ru^2/(n-u)$ and

$$\begin{aligned}
\mathbf{Pr}_\Omega(Y \geq \beta r u) &= \mathbf{Pr}(Y \geq (\beta(n-u)/u)\mathbf{E}Y) \\
&\leq \left(\frac{ue}{\beta(n-u)}\right)^{\beta r u} \qquad \text{by (2)} \\
&\leq \left(\frac{34u}{n}\right)^{\beta r u}.
\end{aligned}$$

As $r \geq 10^6$, $\beta r/2 \gg 1$ and so by Lemma 2

$$\begin{aligned}
\mathbf{Pr}\left(\overline{\mathcal{E}}\right) &\leq e^{2r^2} \sum_{u=2\beta r}^{\theta n} \binom{n}{u} \left(\frac{34u}{n}\right)^{\beta r u} \leq e^{2r^2} \sum_{u=2\beta r}^{\theta n} \left(\frac{ne}{u}\right)^{u} \left(\frac{34u}{n}\right)^{\beta r u} \\
&\leq e^{2r^2} \sum_{u=2\beta r}^{\theta n} \left(\frac{34u}{n}\right)^{\beta r u/2} \leq 2e^{2r^2} \left(\frac{68\beta r}{n}\right)^{\beta^2 r^2} \leq 2\exp\left\{2r^2 - \beta^2 r^2 \log \frac{n}{6r}\right\} \\
&= O(n^{-2}),
\end{aligned}$$

provided $r \leq c_0 n,\ c_0$ sufficiently small. □

Proof of Theorem 1(a). Assume the events $\mathcal{Q}_1, \mathcal{Q}_2$ described in Lemmas 3,4. If G_r is not r-connected then there is a separator X of size $x \leq r-1$. Let $G_r - X = A + B$ and $|A| = a \leq |B| = b$.

Case 1: $2 \leq a \leq r/2$.
Let $u, v \in A$ be arbitrary. If $r = o(n)$ then as $\mathcal{Q}_1$ occurs,

$$|N(u) \cup N(v)| \geq 2r - |N(u) \cap N(v)| \geq 2r - o(r) - 10 \tag{3}$$

However

$$|N(u) \cup N(v)| \leq |A \cup X| \leq a + r - 1 < 3r/2 \tag{4}$$

which contradicts (3).

If $cn \leq r \leq n/4$ for some $c > 0$, we see that because $\mathcal{Q}_1$ occurs we have $|N(u) \cup N(v)| \geq (1 - o(1))7r/4$, which contradicts (4).

Case 2: $r/2 \leq a \leq n/80$.
As $|A \cup X| \leq a + r - 1$ and $A \cup X$ contains at least $ar/2$ edges we see that because $\mathcal{Q}_2$ occurs

$$\frac{ar}{2} \leq \frac{r}{12}(a + r - 1) \text{ and so } a < r/5.$$

Case 3: $n/80 \leq a \leq \lceil n/2 \rceil$.
If configuration F is chosen randomly from Ω then the existence of a separator of size $x \leq r-1$, where the smaller component has size $a \geq n/80$, has probability at most

$$\sum_{a=n/80}^{\lceil n/2 \rceil} \sum_{x=0}^{r-1} \binom{n}{a} \binom{n-a}{x} \left(1 - \frac{b}{n}\right)^{ar/2}.$$

Thus from Lemma 2 the probability of this event in $\mathcal{G}_r$ is at most

$$e^{2r^2} \sum_{a=n/80}^{\lceil n/2 \rceil} 4^n e^{-a(n-(a+r))r/2n} \leq e^{-rn/500} = o(1)$$

for $r \leq c_0 n,\ c_0$ sufficiently small. □

4 Hamilton cycles

We prove Theorem 1(b) on the assumption that $10^7 \le r \le c_0 n$.

Definition: Let $\mathcal{G}_r^*$ denote the subset of $\mathcal{G}_r$ consisting of those graphs G with the following properties:

C1: All sets of vertices U of size at most $n/70$ induce at most $r|U|/12$ edges.

C2: The graph G is connected.

Lemma 4 and Theorem 1(a) imply that

Lemma 5 $|\mathcal{G}_r^*| = (1 - o(1))|\mathcal{G}_r|$.

Given a subset R of the edges of G, let $d_R(v)$ be the number of edges of R which are incident with the vertex v of G.

Definition: Let P be some *fixed* longest path of G. A set of edges $R \subseteq E(G)$ is *deletable* from G, ($R \in \mathrm{Del}(G)$), if

D1: R avoids P.

D2: For all $v \in V$, $\frac{r}{4} \le d_R(v) \le \frac{r}{2}$.

Lemma 6 *Let $G \in \mathcal{G}_r$ and let R be a random subset of the edges of G where each edge of G is placed into R independently with probability 1/3. then*

$$\mathbf{Pr}(R \textit{ is deletable} \mid G) \ge e^{-n}$$

Proof

$$\mathbf{Pr}(D1 \mid G) = \left(\frac{2}{3}\right)^{|P|} \ge \left(\frac{2}{3}\right)^{n} \ge e^{-n/2}.$$

For (D2) we condition on (D1). We use the symmetric version of the Lovász Local Lemma (see for example Alon and Spencer [1]) to show that

$$\mathbf{Pr}(D2 \mid D1) \ge e^{-n/2}.$$

Let A_v be the event $\{d_R(v) \notin [\frac{r}{4}, \frac{r}{2}]\}$, then $\mathbf{Pr}(A_v \mid D1) \le e^{-r/100}$ and the dependency graph has degree at most r. For large r we can apply the lemma to show that conditional on D_1,

$$\mathbf{Pr}\left(\bigcap_{v \in V} \overline{A_v} \mid D1\right) \ge (1 - 2e^{-r/100})^n \ge e^{-n/2}.$$

□

The size of the set R of deleted edges is binomial $B(rn/2, 1/3)$ and thus **whp**$|R| = (1+o(1))rn/6$. For the purposes of Lemma 7 below, we condition on $|R| \in [(.16)rn, (.17)rn]$. We note that there exists some $\delta > 10^{-7}$ such that

$$\mathbf{Pr}\left(|R| \notin [(.16)rn, (.17)rn]\right) \leq e^{-\delta rn}. \tag{5}$$

Definition: A set of edges S is *addable* to a simple graph H, ($S \in \mathrm{Add}(H)$), if

A1: $H + S \in \mathcal{G}_r$.

A2: No longest path of H is closed to a cycle by S.

Let

$$\begin{aligned} \mathcal{N} &= \{G \in \mathcal{G}_r^* : G \text{ is not Hamiltonian } \} \\ \mathcal{E} &= \{(G,R) : G \in \mathcal{N}, R \in \mathrm{Del}(G)\} \\ \Psi &= \{H : H = G - R,\ (G,R) \in \mathcal{E},\ |R| \in [(.16)rn, (.17)rn]\} \\ \mathcal{F} &= \{(G,S) : G \in \mathcal{G}_r, G - S \in \Psi, S \in \mathrm{Add}(G - S)\}. \end{aligned} \tag{6}$$

Remark 4 We note that $\mathcal{E} \subseteq \mathcal{F}$: Let $(G,R) \in \mathcal{E}$ so that $G - R \in \Psi$, and let P be any longest path of G avoided by R. By (C2), G is connected, so P cannot be contained in any cycle, as this would imply either that G was Hamiltonian, or that P was not a longest path. Thus R is addable for $G - R$ and $(G,R) \in \mathcal{F}$.

Lemma 7 *Let $H \in \Psi$. Let $\mathcal{S}(H) = \{S : H + S \in \mathcal{G}_r\}$. Let S be chosen u.a.r from $\mathcal{S}(H)$. There exists a constant $\delta > 10^{-7}$ such that*

$$\mathbf{Pr}(S \in \mathit{Add}(H)) \leq e^{-\delta rn}.$$

Proof

Given y_0 let $P_{y_h} = y_0 y_1 \ldots y_h$ be a longest path starting at y_0 in H. A *Pósa rotation* $P_{y_h} \to P_{y_{i+1}}$, [19, 6] gives the path $P_{y_{i+1}} = y_0 y_1 \ldots y_i y_h y_{h-1} \ldots y_{i+1}$ formed from P_{y_h} by adding the edge $y_h y_i$ and erasing the edge $y_i y_{i+1}$.

Let $END(a)$ be any set of endpoint vertices formed by Pósa rotations with a fixed, of a longest path aPb in H. We prove that $|END(a)| \geq n/210$.

The Pósa condition for the rotation endpoint set U of a longest path P requires that $|N(U)| < 2|U|$, where $N(U)$ is the disjoint neighbour set of U. Let $u = |U|$ and let $\nu = |U \cup N(U)|$. Thus $u > \nu/3$. The condition (D2) guarantees that $U \cup N(U)$ induces at least $ru/4 > r\nu/12$ edges in H. Thus (C1) implies $\nu > n/70$ and $u > n/210$.

Let the degree sequence of R be $\mathbf{d} = (d_1, ..., d_n)$ and that of H be $(r - d_1, ..., r - d_n)$. We choose a replacement set of edges S of size $D = |R| = (d_1 + d_2 + \cdots + d_n)/2$ uniformly among all edge sets with degree sequence $\mathbf{d}$ such that $H + S \in \mathcal{G}_r$. If we generate a random configuration F on $\mathbf{d}$, then conditional on $H + \gamma(F)$ being simple, $\gamma(F) = S$ is a u.a.r element of $\mathcal{S}(H)$.

The probability that $H+\gamma(F)$ is simple.

We generate u.a.r. a configuration F from the set L, size $|L| = 2D$, of configuration points corresponding to the degree sequence $\boldsymbol{d}$, of R. We show that

$$Pr(H + \gamma(F) \text{ is simple}) \geq n^{-2}e^{-4r^2}. \tag{7}$$

We generate the first $rn/12$ random pairings using CONSTRUCT and the rest of F using GENERATE (see Remarks 2, 3). Our reason for this approach is as follows. The ordering $\boldsymbol{\sigma} = (x_1, x_2, ..., x_{2D})$ of L in GENERATE is deterministic. At step $i = 1$, the algorithm GENERATE defaults to Choice A. We cannot ignore the possibility that H already contains the edge $\{\phi(x_1), \phi(x_2)\}$. Similarly, if at step $i + 1$, GENERATE uses Choice B, then as the edges of H are fixed, we cannot argue that the existing edges of F_i avoid neighbours of $\phi(x_1), \phi(x_2)$ in H until $i \gg r^2$.

Assuming that the u_i are chosen randomly for each of the first $rn/12$ iterations, we claim that the probability that CONSTRUCT inserts a loop or parallel edge is at most

$$\frac{r/2 + r^2/2}{(.15)rn} \leq 4r/n.$$

Indeed, when CONSTRUCT starts there are $2D \in [(.32)rn, (.34)rn]$ configuration points to be paired. At the last iteration of CONSTRUCT there are $2D - rn/6 \geq (.15)rn$ points remaining. Each vertex occurs at most $r/2$ times in the sequence (by D2).

CONSTRUCT picks a point u_i and then a random point v_i. Given u_i there are $\leq r/2$ choices which make a loop. In the worst case $d(u_i) = r - 1$ in $H + \gamma(F_{i-1})$ and each neighbour is missing $r/2$ points. This leads to at most $r/2 + r^2/2$ bad choices out of at least $(.15)rn$ choices for v_i.

Let S_1 be the subgraph of S produced by CONSTRUCT. It follows that

$$\mathbf{Pr}(H + S_1 \text{ is simple }) \geq e^{-r^2}.$$

We now continue with GENERATE for the remaining $D - rn/12$ edges to be inserted. The subgraph H remains fixed, and GENERATE is initialized with configuration $F_{rn/12}$ of S_1 on $\{u_1, u_2, ..., u_{rn/6}\}$. For steps $i = rn/12 + 1, \ldots, D$ we run GENERATE with the minimum degree ordering $\boldsymbol{\sigma}$ of $L - \{u_1, u_2, ..., u_{rn/6}\}$ similar to the ordering described in the proof of Lemma 2. Observe that

$$\mathbf{Pr}(H + \gamma(F_i) \text{ is simple} \mid H + \gamma(F_{i-1}) \text{ is simple}) \geq \left(1 - \frac{1}{2i-1}\right)\left(1 - \frac{25r}{n}\right).$$

The probability that the algorithm makes a Type B choice at step i is $1-\frac{1}{2i-1}$. Given a Type B choice, the probability that a loop or multiple edge is formed is at most $25r/n$ for reasons that we now explain. To create a loop we much choose $\phi(z_t)=\phi(x_{2i+t-2})$, for $t=1$ or 2 and there are at most $2r$ choices of $\{z_1,z_2\}$ that will lead to this. To create a parallel edge $\phi(z_t)$ must be a neighbour of $\phi(x_{2i+t-2})$, for $t=1$ or 2 and there are at most $2r^2$ choices of $\{z_1,z_2\}$ that will lead to this. These choices are made randomly from a set of edges of F_i of size at least $rn/12$.

Now $\prod_{i=rn/12+1}^{D}\left(1-\frac{1}{2i-1}\right)\geq n^{-2}$. The number of edges inserted by GENERATE is at most $(.087)rn$ and $\left(1-\frac{25r}{n}\right)^{(.087)rn}\geq e^{-3r^2}$ and so (7) follows.

The probability that $\gamma(F)$ is addable for H.

Let x_0 be an end vertex of longest path P in H. Now let $Y=\{(a,b):\ a\in END(x_0), b\in END(a)\}$. Then $S\in Add(H)$ implies $\gamma(F)\cap Y=\emptyset$. For otherwise the edge ab would close some longest path of H to a cycle.

We will use CONSTRUCT to generate a configuration F with the required degree sequence $(d_1,\ldots,d_n)$.

Since $|END(x_0)|\geq n/210$, the sum of the values d_v over vertices $v\in END(x_0)$ is at least $\frac{r}{4}\frac{n}{210}$. Thus, we can choose u_j so that $\phi(u_j)\in END(x_0)$ for each of the first $\nu=rn/1680$ steps. For $j\leq\nu$, writing a for $\phi(u_j)$, let Y_j be the set of remaining configuration points y such that $\phi(y)\in END(a)$. Then $|Y_j|\geq\frac{r}{4}\frac{n}{210}-2j$. As F contains at most $rn/2$ configuration points,

$$\begin{aligned}\mathbf{Pr}(\gamma(F)\cap Y=\emptyset) &\leq \prod_{j=1}^{\nu}\left(1-\frac{|Y_j|}{rn/2}\right)\\ &\leq \exp\left(-\sum_{j=1}^{\nu}\left(\frac{1}{420}-\frac{4j}{rn}\right)\right)\\ &= e^{-\delta_1 rn}\end{aligned}$$

where $\delta_1\approx 1/(1680\times 840)$.

Thus

$$\mathbf{Pr}(S\in Add(H))\leq e^{-\delta_1 rn}\times n^2e^{4r^2}$$

and the lemma follows.

□

We can now complete the proof of Theorem 1(b). Suppose G is chosen u.a.r. from $\mathcal{G}_r^*$ and then R is chosen by selecting edges independently with probability 1/3. From

Lemma 6, we see that

$$\begin{aligned}\mathbf{Pr}(\mathcal{E}) &= \sum_{G\in\mathcal{N}}\sum_{R\in Del(G)}\mathbf{Pr}((G,R)) \\ &\geq e^{-n}\mathbf{Pr}(\mathcal{N}).\end{aligned}$$

From the definitions (6), inequality (5) and Lemma 7 it follows that

$$\begin{aligned}\mathbf{Pr}(\mathcal{F}) &\leq \mathbf{Pr}(|R|\notin[(.16)rn,(.17)rn]) \\ &+ \sum_{H\in\Psi}\sum_{S\in Add(H)}\mathbf{Pr}((H+S,S)\mid G-R=H)\mathbf{Pr}(G-R=H) \\ &\leq \sum_{H\in\Psi}e^{-\delta rn}\mathbf{Pr}(G-R=H)+e^{-\delta rn} \\ &\leq 2e^{-\delta rn}.\end{aligned}$$

Now, by Remark 4, $\mathcal{E}\subseteq\mathcal{F}$ and so $\mathbf{Pr}(\mathcal{E})\leq\mathbf{Pr}(\mathcal{F})$, thus

$$\mathbf{Pr}(\mathcal{N})\leq 2e^{n-\delta rn}=o(1)$$

and the theorem follows from Lemma 5. □

Remark 5 We note that by following Frieze [10] we can, at the expense of complicating the proof, prove the existence of a polynomial time algorithm for finding a Hamilton cycle.

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