On graph irregularity strength

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Abstract

An assignment of positive integer weights to the edges of a simple graph $G$ is called irregular if the weighted degrees of the vertices are all different. The irregularity strength, $s(G)$, is the maximal weight, minimized over all irregular assignments. In this paper we show, that $s(G) \leq c_1 n/\delta$, for graphs with maximum degree $\Delta \leq n^{1/2}$ and minimum degree $\delta$, and $s(G) \leq c_2 (\log n)n/\delta$, for graphs with $\Delta > n^{1/2}$, where $c_1$ and $c_2$ are explicit constants. To prove the result, we are using a combination of deterministic and probabilistic techniques.

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1 Introduction:

Perhaps the second oldest "fact" in graph theory is that in a simple graph, two vertices must have the same degree. This fact no longer holds for multigraphs. By an irregular multigraph we mean one in which each vertex has a different degree. Hence, a natural question would be: What is the least number of edges we would need to add to a graph in order to convert a simple graph into an irregular multigraph?

Another way to view this question is via an assignment of integer weights to the edges of the graph. Given a simple graph $G$ of order $n$, an assignment $f : E(G) \to \{1, ..., w\} = |w|$ of positive integers weights to the edges of $G$ is called irregular if the weighted degrees, $f(v) = \sum_{u \in N(v)} f(uv)$ of the vertices are all different. The irregularity strength, $s(G)$, is the maximal weight $w$, minimized over all irregular weight assignments, and is set to $\infty$ if no such assignment is possible. Clearly, $s(G) < \infty$ if and only if $G$ contains no isolated edges and at most one isolated vertex.

The irregularity strength was introduced in [3] by Chartrand et al.. The irregularity strength of regular graphs was considered by Faudree and Lehel in [4]. They showed that if $G$ is a $d$-regular graph of order $n$, $d \geq 2$, then $s(G) \leq \lceil n/2 \rceil + 9$, and they conjectured that $s(G) = \left\lceil \frac{n+d-1}{d} \right\rceil + c$ for some constant $c$. This conjecture comes from the lower bound $s(G) \geq \left\lceil \frac{n+d-1}{d} \right\rceil$. For general graphs with finite irregularity strength, Aigner and Triesch [1] showed that $s(G) \leq n - 1$ if $G$ is connected and $s(G) \leq n + 1$ otherwise. Nierhoff [8] refined their method to show $s(G) \leq n - 1$ holds for all graphs with finite irregularity strength, except for $K_3$. We will provide an improvement of both the Faudree-Lehel bound and the Aigner-Triesch-Nierhoff bound in this paper.

For a review of other results and open problems in this area we refer the reader to a survey paper by Lehel [7].

In this paper all graphs are simple of order $n$. The degree of a vertex $v$ is denoted by $d_v$ or $\text{deg}(v)$, we shall denote the minimum degree of $G$ by $\delta$ and the maximum degree by $\Delta$. For terms not found here see [2] or [6]. Our upper bounds on $s(G)$ involve a function of $n$ and $\delta$ or both $\delta$ and $\Delta$, and are stated in the next Theorem.
Theorem 1 Let $G$ be a graph with no isolated vertices or edges.

(a) If $\Delta \leq \lfloor (n/\ln n)^{1/4} \rfloor$, then $s(G) \leq 7n\left(\frac{1}{\delta} + \frac{1}{\Delta}\right)$.

(b) If $\lfloor (n/\ln n)^{1/4} \rfloor + 1 \leq \Delta \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 60n/\delta$.

(c) If $\Delta \geq \lfloor n^{1/2} \rfloor + 1$, $\delta \geq \lfloor 6 \log n \rfloor$ then $s(G) \leq 336(\log n)n/\delta$.

For regular graphs, we get the following Theorem with improved constants.

Theorem 2 Let $G$ be a $d$-regular graph with no isolated vertices or edges.

(a) If $d \leq \lfloor (n/\ln n)^{1/4} \rfloor$, then $s(G) \leq 10n/d + 1$.

(b) If $\lfloor (n/\ln n)^{1/4} \rfloor + 1 \leq d \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 48n/d + 1$.

(c) If $d \geq \lfloor n^{1/2} \rfloor + 1$, then $s(G) \leq 240(\log n)n/d + 1$.

Observe that both (a) and (b) give bounds of the correct order of magnitude. If $\Delta \geq \lfloor n^{1/2} \rfloor + 1$ and $\delta < \lfloor 6\ln n \rfloor$, Theorem 1 does not apply, but we can still make the following statement:

Theorem 3 Let $G$ be a graph with no isolated vertices or edges. If $n$ is sufficiently large, then $s(G) \leq 14n/\delta^{1/2}$.

To explain the main technique used to prove all results let us define

$$m_g = \max_{X \subseteq V(G)} \{|X| : g(v) = g(u) \text{ for all } v, u \in X\},$$

where $g$ is defined as a weight assignment, i.e., $g : E(G) \to \{1, 2, \ldots, w\} = [w]$, for some integer $w$. In the deterministic part of our proof (see Lemma 4) we show that $s(G) \leq 3(w+1)m_g$. Next, we use probabilistic tools to establish bounds on $m_g$. Here the idea is to assign weights to edges from the set $\{1, 2\}$ or $\{1, 2, 3\}$, and show that for such weightings, there exist assignments with $m_g$ of the order $n/\delta$ or $n \log n/\delta$ (see Lemmas 7, 8 and 9).
2 Deterministic Lemmas

The next two Lemmas will be fundamental to our results. Their proofs follow below.

**Lemma 4** Let $G$ be a graph without isolated vertices or isolated edges. Let $g : E(G) \to [w]$ be a weight assignment. Then there exists an irregular assignment $f : E(G) \to \{2m_g, \ldots, (3w + 1)m_g\}$.

**Lemma 5** Let $G$ be a $d$-regular graph without isolated vertices or isolated edges. Let $g : E(G) \to [w]$ be a weight assignment. Then there exists an irregular assignment $f : E(G) \to [(3w - 1)m_g + 1]$.

We begin with a lemma needed to prove Lemma 4. We will call a tree with at most one vertex of degree greater than two, and $k$ vertices of degree one, a generalized $k$-star.

**Lemma 6** Let $G$ be a graph without isolated vertices or isolated edges. Then $G$ has a factor consisting of generalized stars of order at least three.

**Proof:** Let $T$ be a spanning tree of a component of $G$. Note that $|V(T)| \geq 3$ by our hypothesis. We show that $T$ can be broken into disjoint generalized stars that together span $V(T)$. Then repeating this argument on each component produces the result.

To do this we induct on $|U|$, where $U = \{u \in V(T) \mid \deg_T(u) \geq 3\}$. If $|U| \leq 1$ we are done, as $T$ is itself a generalized star. Now assume the result holds on any tree $T$ with $|U| = l \geq 1$ and suppose $T$ is a tree with $|U| = l + 1$. Now root $T$ at $u \in U$ and select any vertex $v \in U, v \neq u$, such that the distance in $T$ between $u$ and $v$ is maximum over all vertices of $U$. Let $T_v$ be the subtree of $T$ rooted at $v$ and consider $T' = T \setminus T_v$. This tree has $|U| = l$ and by the induction hypothesis, we can find generalized stars in $T'$ that span $V(T')$. Further, the tree $T_v$ is, by our choice of $v$, a generalized star of order at least three. This star, together with the collection of stars that spans $T'$, spans $T$, completing the proof.

**Proof of Lemma 4.** Denote the weight class of a vertex $v \in V(G)$ as

$$C_v = \{u \in V(G) : g(u) = g(v)\}.$$
Define a new weight function \( \hat{f} : E \to [3m_g w] \) by \( \hat{f}(e) = 3m_g g(e) \). Note that the weight classes are unchanged under this function. Let \( S \) be a generalized star factor of \( G \), guaranteed by Lemma 6. We select one generalized star \( S \) from \( S \). Let \( u \) be a vertex of maximum degree in \( S \) and suppose that \( S \) consists of \( t \) paths rooted at \( u \). Let \( u_1, u_2, \ldots, u_t \) be the neighbors of \( u \) in \( S \). Consider the first branch (path) of \( S \), say \( v_1, v_2, \ldots, v_r \), where \( v_1 = u_1 \) and \( r \geq 2 \) (if such a branch of \( S \) exists). Now begin with the last edge \( v_r v_{r-1} \). We change the weight of this edge as follows. Put \( f(v_r v_{r-1}) = \hat{f}(v_r v_{r-1}) + x \), where \( x \) is selected from the set \( L = \{0, -1, \ldots, -(m_g - 1)\} \) in such a way that \( f(v_r) \), its new weighted degree, is different from the current weighted degrees of any vertex from \( C_{v_r} \setminus \{v_r\} \). Since \( |C_{v_r}| \leq m_g \), it is always possible to select an appropriate \( x \). We now repeat this process to the edges \( v_{r-1} v_{r-2}, v_{r-2} v_{r-3}, \ldots v_2 v_1 \), thus making \( f(v_{r-1}), f(v_{r-2}), \ldots, f(v_2) \) unique also. To complete the first phase, repeat the procedure on the paths emanating from \( u_2, u_3, \ldots, u_t \), in this order.

It remains to adjust the weights of the star centered at \( u \). So, we change the weights of the edges \( uu_1, uu_2, \ldots, uu_{t-1} \), one by one, starting at \( uu_1 \). Let \( f(uu_i) = \hat{f}(uu_i) + y_i \), where \( y_i \) is chosen from the set \( L' = \{-m_g, -(m_g - 1), \ldots, m_g - 1, m_g\} \), in such a way that \( f(u_i), i = 1, 2, \ldots, t - 1 \), the new weighted degree of \( u_i \), is different from the current weighted degrees of any vertex from \( C_u \setminus \{u_i\} \) and, additionally, such that \( \sum_{i=1}^t y_i \) belongs to the set \( (L' \cup \{-m_g\}) \setminus \{f(u_{t-1}v) - \hat{f}(u_{t-1}v)\} \), where \( v \) is the second vertex of the path starting in \( u_t \) (if no such vertex \( v \) exists, use instead \( (L' \cup \{-m_g\}) \setminus \{0\} \)). Now we are left with \( uu_t \). Observe that \( u \) and \( u_t \) have different weighted degrees at this time. Now let \( f(uu_t) = \hat{f}(uu_t) + x \), where \( x \in L' \setminus \{-m_g\} \), such that both \( f(u) \) and \( f(u_t) \) are unique in their respective classes. This is possible, since there are \( 2m_g \) options, and \( C_u \) and \( C_{u_t} \) can only block \( 2(m_g - 1) \) of these. Finally, repeat the process for all remaining stars \( S \in S \).

Now for every weight class \( C_u \), all vertices have different weighted degrees under \( f \). The weighted degrees were altered from \( \hat{f} \) by total values from the range \( \{-2m_g + 1, \ldots, m_g\} \), the different classes were at least \( 3m_g \) apart from each other under \( \hat{f} \), so \( f \) is an irregular assignment to the set \( \{2m_g, 2m_g + 1, \ldots, 3m_g w + m_g\} \).

**Proof of Lemma 5.** Use Lemma 4 to get an irregular weight assignment \( f' : E(G) \to \{2m_g, 2m_g + 1, \ldots, 3m_g w + m_g\} \). Now define \( f : E(G) \to [(3w - 1)m_g + 1] \) by \( f(e) = f'(e) - 2m_g + 1 \). This assignment is irregular,
since the weighted degree of every vertex is reduced by $d(2m_g - 1)$.

3 Probabilistic Lemmas

The following two lemmas will be used to get bounds on the irregularity strength of graphs with maximal degree $\Delta \leq n^{1/2}$. Again, the proofs follow below.

Lemma 7 Let $G$ be a graph. If $\Delta \leq (n/\ln n)^{1/4}$, then $\exists g : E(G) \to \{1, 2\}$ such that $m_g \leq \frac{n}{\delta} + \frac{n}{\Delta}$.

Lemma 8 Let $G$ be a graph. If $\Delta \leq n^{1/2}$, then $\exists g : E(G) \to \{1, 2, 3\}$ such that $m_g \leq 6n/\delta$.

The next lemma is used for graphs with $\Delta > n^{1/2}$.

Lemma 9 Let $G$ be a graph. If $n \geq 10$ and $\delta \geq 10 \log n$, then $\exists g : E(G) \to \{1, 2\}$ such that $m_g \leq 48(\log n)n/\delta$.

Finally, we state the lemma which provides bounds on $m_g$, without any restrictions on vertex degrees of a graph $G$, but for sufficiently large $n$ only.

Lemma 10 Let $G$ be a graph. If $n$ is sufficiently large, then $\exists g : E(G) \to \{1, 2\}$ such that $m_g \leq 2n/\delta^{1/2}$.

Since the proofs of both Lemma 7 and Lemma 9 use the same model of assigning weights to the edges, at random, we will present their proof together.

Proof of Lemmas 7 and 9.
Let $X_v, v \in V$ be independent random variables with uniform distribution over the interval $[0, 1]$, and then for $e = uv \in E$, let

$$g(e) = \begin{cases} 
2 & \text{if } X_u + X_v \geq 1 \\
1 & \text{if } X_u + X_v < 1.
\end{cases}$$

For the non-negative integer $y \in \{0, 1, \ldots, d_v\}$,

$$\Pr(g(v) = d_v + y) = \int_{x=0}^{1} \binom{d_v}{y} x^y (1 - x)^{d_v - y} dx = \frac{1}{d_v + 1} - \frac{1}{\delta + 1}. \quad (1)$$

6
It follows for every $y$ with $\delta \leq y \leq 2\Delta$ and $Z_y = |\{v \in V : g(v) = y\}|$ that
\[
E(Z_y) \leq \frac{n}{\delta + 1}.
\] (2)

To prove Lemma 7, we assume that $G$ is a graph with maximum degree
\[
\Delta \leq (n/\log n)^{1/4}.
\]

We apply the Hoeffding-Azuma inequality, see e.g. Janson, Łuczak and
Ruciński [6]. Changing the value of an $X_v$ can only change the value of $Z_y$
by at most $\Delta + 1$. It follows that for $t > 0$,
\[
Pr(Z_y \geq E(Z_y) + t) \leq \exp \left\{ -\frac{t^2}{2n(\Delta + 1)^2} \right\}.
\] (3)

Putting $t = \frac{n}{\Delta + 1}$ and using (2) we see that
\[
Pr(Z_y \geq E(Z_y) + t) < \frac{1}{2\Delta},
\]
and thus
\[
Pr(\exists y : Z_y \geq \frac{n}{\delta} + \frac{n}{\Delta}) < 1,
\]
and Lemma 7 follows.

We now prove Lemma 9. We use the Markov inequality for $t, k > 0$ and
any event $\mathcal{E}$, to obtain
\[
Pr(Z_y > t \mid \mathcal{E}) \leq \frac{E\left(\left(\frac{Z_y}{k}\right) \mid \mathcal{E}\right)}{\left(\begin{array}{c} t \\ k \end{array}\right)}.
\] (4)

But
\[
E\left(\left(\frac{Z_y}{k}\right) \mid \mathcal{E}\right) = \sum_{|S| = k} Pr(g(v) = y, v \in S \mid \mathcal{E}) = \sum_{|S| = k} Pr(g(v) = y, v \in S \mid \mathcal{E}).
\] (5)

Now fix $S = \{v_1, v_2, \ldots, v_k\}$ in (5). For $v \in S$ let $N_S(v) = N(v) \setminus S$, and let $\mu(v) = |N_S(v)|$. Note that $d_v - \mu(v) \leq k - 1$. For $v \in S$ let $\xi_1 < \xi_2 < \cdots < \xi_d_v$ be the values of $X_v, u \in N(v)$, sorted in increasing order and let $\eta_1 < \eta_2 < \cdots < \eta_{\mu(v)}$ be the values of $X_u, u \in N_S(v)$, also sorted in increasing order.
Note that, in general, if \( \xi_1 < \xi_2 < \cdots < \xi_s \) is the sequence of order statistic from the uniform distribution over \([0, 1]\), then \( \xi_i \) has the same distribution as \((Y_1 + Y_2 + \cdots + Y_i)/(Y_1 + Y_2 + \cdots + Y_{s+1})\) where \( Y_1, Y_2, \ldots, Y_{s+1} \) is a sequence of independent random variables, each having exponential distribution with mean one, see for example Ross, Theorem 2.3.1 [9].

To prove the lemma we need to show the following general statement.

**Lemma 11** Let \( Y_1, Y_2, \ldots, Y_s \) be a sequence of independent random variables, each having exponential distribution with mean one. Then for any real \( a > 0, 0 < b < 1 \) we have

\[
\Pr(Y_1 + \ldots + Y_s \geq (1 + a)s) \leq ((1 + a)e^{-a})^s
\]

\[
\Pr(Y_1 + \ldots + Y_s \leq (1 - b)s) \leq ((1 - b)e^b)^s.
\]

**Proof:**

\[
\Pr(Y_1 + \ldots + Y_s \geq t) \leq \Pr(e^{\lambda(Y_1 + \cdots + Y_s - t)} \geq 1)
\]

\[
\leq e^{-\lambda t} E(e^{\lambda(Y_1 + \cdots + Y_s)})
\]

\[
= e^{-\lambda t}(1 - \lambda)^s,
\]

provided \( \lambda \in (0, 1) \).

So putting \( t = (1 + a)s \), we see that

\[
\Pr(Y_1 + \ldots + Y_s \geq (1 + a)s) \leq \left( \frac{e^{-\lambda(1+a)}}{1 - \lambda} \right)^s
\]

\[
= ((1 + a)e^{-a})^s
\]

on putting \( \lambda = a/(1 + a) \).

A similar argument shows that

\[
\Pr(Y_1 + \ldots + Y_s \leq (1 - b)s) \leq ((1 - b)e^b)^s,
\]

completing the proof of Lemma 11.

Let \( k = \lfloor \log n \rfloor \) and

\[
\mathcal{E} = (\Theta < (16 \log n)/\delta),
\]

where

\[
\Theta = \max_{v \in V} \Theta_v, \quad \text{and} \quad \Theta_v = \max_{0 \leq i \leq d_v - 2k + 1} \xi_{i+2k} - \xi_i.
\]

Here, by default, we take \( \xi_0 = 0 \) and \( \xi_{d_v+1} = 1 \).
Now, observe that \( g(v) = y \) implies
\[
1 - X_v \in [\xi_{2d_v - y}, \xi_{2d_v - y + 1}] \subseteq [\eta_{2d_v - y - k + 1}, \eta_{2d_v - y + 1}] \subseteq [\xi_{2d_v - y - k + 1}, \xi_{2d_v - y + k}].
\]
In the above formula, we take \( \xi_j = \eta_j = 0 \) for \( j \leq 0 \), and \( \xi_{d_v + j} = \eta_{\mu(v) + j} = 1 \) for \( j \geq 1 \).

Applying Lemma 11 to the order statistics defining \( \Theta \), we see that
\[
\Pr(\neg \mathcal{E}) = \Pr \left( \exists v \in V : \Theta_v \geq \frac{16 \log n}{\delta} \right)
\leq n \Pr \left( \exists 0 \leq i \leq \Delta - 2k + 1 : \frac{Y_i + \cdots + Y_{i+2k-1}}{Y_1 + \cdots + Y_{\delta+1}} \geq \frac{16 \log n}{\delta} \right)
\leq n \Pr(Y_1 + \cdots + Y_{\delta+1} \leq \delta/2) + n^2 \Pr(Y_1 + \cdots + Y_{2k} \geq 8k)
\leq n \left( \frac{1}{2} \right)^{\delta+1} + n^2 \left( \frac{4e^{-3}}{2} \right)^{2k}
\leq \frac{1}{10}.
\]

Further,
\[
\Pr(\neg \mathcal{E}) \leq \Pr(1 - X_v \in [\eta_{2d_v - y - k + 1}, \eta_{2d_v - y + 1}], i = 1, 2, \ldots, k | \mathcal{E})
\leq 2 \Pr(1 - X_v \in [\eta_{2d_v - y - k + 1}, \eta_{2d_v - y - k + 1} + \frac{16 \log n}{\delta}], i = 1, 2, \ldots, k)
\leq 2 \left( \frac{16 \log n}{\delta} \right)^k.
\]

From (4) and (5) we obtain
\[
\Pr(\exists y : Z_y > t | \mathcal{E}) \leq 2n \left( \frac{t}{k} \right)^{-1} \left( \frac{n}{k} \right) \left( \frac{16 \log n}{\delta} \right)^k.
\]

Putting \( t = 48(\log n)n\delta^{-1} \) together with (6) establishes
\[
\Pr(\exists y : Z_y > t) \leq \Pr(\exists y : Z_y > t | \mathcal{E}) + \Pr(\neg \mathcal{E}) < 1,
\]
proving Lemma 9.
Proof of Lemma 8. For every vertex $v$ independently assign a number $W_v$ from $\{0, \ldots, d_v\}$ uniformly at random. Now pick a random subset $N \subseteq N(v)$ of size $W_v$, and for every $u \in N$, set $v_u = 1$, and for every $u \in N(v) \setminus N$, set $v_u = 0$. 

Let $g : E \to [3]$ as follows: For $uv \in E$, let $g(uv) = 1 + v_u + u_v$. For a vertex $v$, let $g(v) = \sum_{u \in N(v)} g(uv)$. For some integer $y$ with $\delta \leq y \leq 3\Delta$, let $Z_y = \{v \in V : g(v) = y\}$. Then

$$E(Z_y) \leq \frac{n}{\delta},$$

(7)

since

$$Pr(g(v) = y) = Pr(W_v = y - d - \sum_{u \in N(v)} u_v) \leq \frac{1}{d_v + 1}.$$

By the symmetry of the construction we know that $\forall x \in V, v, u \in N(x)$:

$$Pr(x_v = 1) = 1/2,$$

$$Pr(x_v = x_u = 1) = Pr(x_v = x_u = 0) = 1/3,$$

$$Pr(x_v = 1, x_u = 0) = Pr(x_v = 0, x_u = 1) = 1/6.$$ 

(8)

To use Chebyshev’s inequality, we have to bound the variance of $Z_y$:

$$Var(Z_y) = \sum_{v \in V} \sum_{u \in V} (Pr(g(v) = g(u) = y) - Pr(g(v) = y)Pr(g(u) = y)).$$

Fix a $v \in V$, and consider

$$S_v = \sum_{u \in V} (Pr(g(v) = g(u) = y) - Pr(g(v) = y)Pr(g(u) = y)).$$

Divide $V$ into three classes $V_1, V_2, V_3$, and consider the partial sums

$$S_i = \sum_{u \in V_i} (Pr(g(v) = g(u) = y) - Pr(g(v) = y)Pr(g(u) = y)).$$

Class 1: $V_1 = \{v\}$.

$$S_1 \leq Pr(g(v) = y) \leq \frac{1}{d_v} \leq \frac{\Delta}{\delta^2},$$

(9)
Class 2: $V_2 = N(v)$.

$$S_2 \leq d_v \Pr(g(v) = g(u) = y)$$

$$\leq d_v \Pr \left( W_v = y - d_v - \sum_{x \in N(v)} x_v \mid g(u) = y \right) \Pr \left( W_u = y - d_u - \sum_{x \in N(u)} x_u \right)$$

$$\leq d_v \frac{2}{(d_v + 1)(d_u + 1)} < \frac{2\Delta}{\delta^2}.$$ (10)

Class 3: $V_3 = V \setminus (\{v\} \cup N(v))$.

Let $u \in V_3$, and let $c = |N(v) \cap N(u)|$. For the sake of the analysis, pick a random subset $A$ from $\{x \in N(u) \cap N(v) : x_u = x_v\}$, by choosing each vertex with probability $1/2$. So, using (8), for every vertex $x \in N(u) \cap N(v)$,

$$\Pr(x_u = x_v = 1 \land x \in A) = \Pr(x_u = x_v = 1 \land x \notin A) =$$

$$\Pr(x_u = x_v = 0 \land x \in A) = \Pr(x_u = x_v = 0 \land x \notin A) =$$

$$\Pr(x_u = 0 \land x_v = 1) = \Pr(x_u = 1 \land x_v = 0) = 1/6,$$

and

$$\Pr(x \in A) = 1/3.$$

Let $A \subseteq N(u) \cap N(v)$, and let $a = |A|$.

Then, for every vertex $x \in N(u) \cap N(v)$,

$$\Pr(x_u = x_v = 1 \mid A = A \land x \notin A) = \frac{\Pr(x_u = x_v = 1 \land A = A \mid x \notin A)}{\Pr(A = A \mid x \notin A)} =$$

$$\frac{1/6}{(1/3)(1/3)^a(2/3)^{c-a-1}} = \frac{1}{4}.$$

By symmetry, we get

$$\Pr(x_u = x_v = 0 \mid A = A) = \Pr(x_u = 0, x_v = 1 \mid A = A) =$$

$$\Pr(x_u = 1, x_v = 0 \mid A = A) = 1/4.$$

Thus, given $x \notin A$ and $A = A$, the events $(x_v = 1)$ and $(x_u = 1)$ are independent. For $x \in A$, we get

$$\Pr(x_u = x_v = 1 \mid A = A \land x \in A) = \Pr(x_u = x_v = 0 \mid A = A \land x \in A) = 1/2.$$
We introduce the following notation:

\[
P_A = \Pr(g(v) = g(w) = y \mid A = A) - \Pr(g(v) = y \mid A = A)\Pr(g(w) = y \mid A = A)
\]

\[
= \Pr(g(v) = g(w) = y \mid A = A) - \Pr(g(v) = y)\Pr(g(w) = y),
\]

since \(\Pr(g(v) = y)\) is independent from the choice of \(A\). In particular,

\[
P_\emptyset = \Pr(g(v) = g(w) = y \mid A = \emptyset) - \Pr(g(v) = y)\Pr(g(w) = y) = 0.
\]

(11)

For \(A \neq \emptyset\), pick any \(x \in A\). We want to bound the difference \(P_A - P_{A\setminus x}\). Let

\[
b_v = d_v + \sum_{z \in N(v) \setminus x} z_v, \quad b_u = d_u + \sum_{z \in N(u) \setminus x} z_u.
\]

Now consider the difference between \(P_A\) and \(P_{A\setminus x}\), given that \(b_v = l\) and \(b_u = r\), and denote it by

\[
P_{l,r}^{A\setminus x} - p_{l,r}^{A\setminus x} =
\]

\[
= \Pr(g(v) = g(w) = y \mid A = A \land b_v = l \land b_u = r)
\]

\[
- \Pr(g(v) = g(w) = y \mid A = A \setminus x \land b_v = l \land b_u = r)
\]

\[
= [\Pr(x_u = x_v = 1 \mid A = A) - \Pr(x_u = x_v = 1 \mid A = A \setminus x)]
\]

\[\times \Pr(W_v = y - l - 1)\Pr(W_u = y - r - 1)
\]

\[
+ [\Pr(x_u = x_v = 0 \mid A = A) - \Pr(x_u = x_v = 0 \mid A = A \setminus x)]
\]

\[\times \Pr(W_v = y - l)\Pr(W_u = y - r)
\]

\[
+ [\Pr(x_u = 1 \land x_v = 0 \mid A = A) - \Pr(x_u = 1 \land x_v = 0 \mid A = A \setminus x)]
\]

\[\times \Pr(W_v = y - l - 1)\Pr(W_u = y - r - 1)
\]

\[
+ [\Pr(x_u = 0 \land x_v = 1 \mid A = A) - \Pr(x_u = 0 \land x_v = 1 \mid A = A \setminus x)]
\]

\[\times \Pr(W_v = y - l)\Pr(W_u = y - r)
\]

\[
= \frac{1}{4} [\Pr(W_v = y - l - 1)\Pr(W_u = y - r - 1) + \Pr(W_v = y - l)\Pr(W_u = y - r)
\]

\[- \Pr(W_v = y - l)\Pr(W_u = y - r - 1) - \Pr(W_v = y - l - 1)\Pr(W_u = y - r)].
\]

Therefore,

\[
P_{l,r}^{A\setminus x} - p_{l,r}^{A\setminus x} = \begin{cases} 
1/[4(d_v + 1)(d_u + 1)] & \text{if } (r = y - d_u - 1 \land l = y - d_v - 1) \\
-1/[4(d_v + 1)(d_u + 1)] & \text{if } (r = y - d_u - 1 \land l = y) \\
0 & \text{otherwise.}
\end{cases}
\]
Thus, summing over all possible values of \( l, r \) and 
\( t = \{ z \in A \mid x : z_u = z_v = 1 \} \),

\[
P_A - P_{A\wedge x} \leq \\
\leq \frac{1}{4(d_v + 1)(d_u + 1)} \left[ \Pr(b_u = y - d_u - 1 \land b_v = y - d_v - 1) + \Pr(b_u = y \land b_v = y) \right] \\
\leq \frac{1}{4(d_v + 1)(d_u + 1)} \left[ \sum_{t=0}^{a-1} \binom{a-1}{t} 2^{-a+1} \left( \frac{d_u - a}{y - 2d_u - 1 - t} \right) \left( \frac{d_v - a}{y - 2d_v - 1 - t} \right) 2^{-d_u - d_v + 2a} \\
+ \sum_{t=0}^{a-1} \binom{a-1}{t} 2^{-a+1} \left( \frac{d_u - a}{y - d_u - t} \right) \left( \frac{d_v - a}{y - d_v - t} \right) 2^{-d_u - d_v + 2a} \right] \\
\leq \frac{1}{(d_v + 1)(d_u + 1)} \left( \frac{d_u - a}{(d_u - a)/2} \right) \left( \frac{d_v - a}{(d_v - a)/2} \right) 2^{-d_u - d_v + a} \sum_{t=0}^{a-1} \left( \frac{a - 1}{t} \right).
\]

Suppose first that \( 1 \leq a \leq \delta/3 \). Then,

\[
P_A - P_{A\wedge x} \leq \frac{2^{-d_u - d_v + 2a - 1}}{(d_v + 1)(d_u + 1)} \left( \frac{2^{d_u - a + 1}}{(d_v - a)^{1/2}} \right) \left( \frac{2^{d_v - a + 1}}{(d_u - a)^{1/2}} \right) = \\
\leq \frac{2}{(d_v + 1)(d_u + 1)(d_v - a)^{1/2}(d_u - a)^{1/2}} \leq \frac{3}{d_v \delta^2}.
\]

Hence,

\[
P_A \leq \frac{3a}{d_v \delta^2} \leq \frac{3c}{d_v \delta^2}.
\]  \( (12) \)

Note that for all \( A \),

\[
\Pr(g(v) = g(u) = y \mid A = A) \leq \frac{1}{(d_v + 1)(d_u + 1)},
\]

hence, for \( a > \delta/3 \),

\[
P_A \leq \Pr(g(v) = g(u) = y \mid A = A) \leq \frac{3a}{d_v \delta^2} \leq \frac{3c}{d_v \delta^2}.
\]  \( (13) \)

Therefore, combining (11), (12) and (13),

\[
\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y) \leq \\
\sum_{A \subseteq N(u) \cap N(v)} (3c/d_v \delta^2) \Pr(A = A) = \frac{3|N(v) \cap N(u)|}{d_v \delta^2}.
\]
Now notice that $\sum_{u \in V} |N(v) \cap N(u)|$ counts the number of walks of length two starting in $v$, thus $\sum_{u \in V} |N(v) \cap N(u)| \leq d_v \Delta$, and therefore

$$S_3 \leq \sum_{u \in V_3} \frac{3|N(v) \cap N(u)|}{d_v \delta^2} \leq \frac{3 \Delta}{\delta^2}. \quad (14)$$

Altogether, we get from (9), (10) and (14),

$$S_v = S_1 + S_2 + S_3 \leq \frac{6 \Delta}{\delta^2},$$

and thus,

$$\text{Var}(Z_y) = \sum_{v \in V} S_v \leq \frac{6n \Delta}{\delta^2}.$$

By Chebyshev’s inequality and (7) we get

$$\Pr(Z_y > 6n/\delta) \leq \frac{\text{Var}(Z_y)}{(5n/\delta)^2} < \frac{1}{3 \Delta},$$

and thus,

$$\Pr(\exists y : Z_y > 6n/\delta) < 1,$$

finishing the proof.

Proof of Lemma 10.

Choose $g$ randomly from $\{1, 2\}^E$. Observe that $g(v) - d_v$ has the binomial distribution $Bi(d_v, 1/2)$. For a non-negative integer $y$ let

$$V_y = \{v : |y - \frac{3}{2} d_v| \leq (2d_v \log n)^{1/2}\}.$$

The Chernoff bounds for the tails of the binomial (see for example [6]) imply that for any $t > 0$,

$$\Pr(|g(v) - \frac{3}{2} d_v| \geq t) \leq e^{-2t^2/d_v}.$$

Hence,

$$\Pr(g(v) = y) \leq \frac{1}{n^4} \quad \text{if } v \notin V_y. \quad (15)$$
Now consider \( v \in V_y \). Clearly,
\[
\Pr(g(v) = y) = 0 \quad \text{if } d_v < y/2. \tag{16}
\]

**Case 1: \( y \geq n^{1/4} \)**

If \( d_v \geq y/2 \geq n^{1/4}/2 \) then we can use Stirling’s inequality or apply Feller [5], Chapter VII (2.7) to get
\[
\Pr(g(v) = y) = \frac{1}{2^{d_v}} \binom{d_v}{y - d_v} \approx \sqrt{\frac{2}{\pi d_v}} e^{-z^2/2}, \tag{17}
\]
where \( z = 2(y - \frac{3}{2}d_v)/d_v^{1/2} \).

Let \( Z_y = |\{v : g(v) = y\}| \). It follows from (15), (16) and (17) that
\[
E(Z_y) \leq \frac{|V_y|}{\delta^{1/2}}. \tag{18}
\]

Let
\[
Z_y^1 = |\{v \in V_y : g(v) = y\}| \text{ and } Z_y^2 = |\{v \notin V_y : g(v) = y\}|.
\]

It follows from (15) that
\[
\Pr(Z_y^2 \neq 0) \leq \frac{1}{n^3}. \tag{19}
\]

Note also that \( v \in V_y \) implies that
\[
y = \frac{3}{2}d_v + O \left( (d_v \log n)^{1/2} \right). \tag{20}
\]

Now for \( t > 0 \) and \( k = (\log n)^2 \) we use the Markov inequality to obtain
\[
\Pr(Z_y^1 > t) \leq \frac{E\left(\binom{Z_y^1}{k}\right)}{t}. \tag{21}
\]

But
\[
E\left(\binom{Z_y^1}{k}\right) = \sum_{S \subseteq V_y, |S| = k} \Pr(g(v) = y, v \in S)
\]
\[
= \sum_{S \subseteq V_y, |S| = k} \sum_{\xi \in \{1,2\}^E} \Pr(g(v) = y, v \in S \mid g(E_S) = \xi) \Pr(g(E_S) = \xi) \tag{22}
\]
where \( E_S = \{ e \in E : e \subseteq S \} \).

Now fix \( S \) in (22). For \( v \in S \) let

\[
A_v = \{ e = uv \in E : u \notin S \} \quad \text{and} \quad B_v = \{ e = uv \in E : u \in S \}.
\]

Then, if \( |g(B_v)| \) denotes \( \sum_{u \in B_v} g(u) \),

\[
\Pr(g(v) = y \mid g(E_S) = \xi) = \Pr(|g(A_v)| = y - |g(B_v)|) = 2^{-|A_v|} \left( y - |g(B_v)| - |A_v| \right).
\]

Therefore,

\[
\frac{\Pr(|g(A_v)| = y - |g(B_v)|)}{\Pr(g(v) = y)} = 2^{[B_v] \left( \frac{|A_v|}{y - |g(B_v)| - |A_v|} \right)}
\]

\[
= 2^{[B_v]} \left[ (|A_v| - 1) \cdots (2|A_v| + |g(B_v)| - y + 1) \right] \frac{1 \times 2 \times \cdots \times (y - |g(B_v)| - |A_v|)}{d_v (d_v - 1) \cdots (2d_v - y + 1)}.
\]

Now we use

\[
|A_v| + |B_v| = d_v \quad \text{and} \quad |B_v| \leq |g(B_v)| \leq 2|B_v| \leq 2k
\]

and (20) to verify that

\[
\frac{1 \times 2 \times \cdots \times (y - d_v)}{1 \times 2 \times \cdots \times (y - |g(B_v)| - |A_v|)} =
\]

\[
(y - d_v)(y - d_v - 1) \cdots (y - |g(B_v)| - |A_v| + 1) = \left( \frac{1}{2d_v} \right)^{|g(B_v)|-|B_v|} \left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right)
\]

(25)

and

\[
\frac{|A_v|(|A_v| - 1) \cdots (2|A_v| + |g(B_v)| - y + 1)}{d_v (d_v - 1) \cdots (2d_v - y + 1)} =
\]

\[
\frac{(2d_v - y)(2d_v - y - 1) \cdots (2|A_v| + |g(B_v)| - y + 1)}{d_v (d_v - 1) \cdots (|A_v| + 1)} =
\]

\[
d_v^{B_v - |g(B_v)|} \times 2^{[g(B_v)]-2|B_v|} \left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right).
\]

(26)
Plugging (25) and (26) into (24) we see that
\[
\frac{\Pr(|g(A_v)| = y - |g(B_v)|)}{\Pr(g(v) = y)} = 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right).
\]
So from (22) and (23) we see that
\[
\mathbb{E} \left( \binom{Z^1_y}{k} \right) \leq \\
\sum_{S \subseteq V_y | |S| = k} \sum_{\xi \in \{1, 2\}^{E_S}} \prod_{v \in S} \left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right) \Pr(g(v) = y) \Pr(g(E_S) = \xi) \\
\leq \left( 1 + O \left( k^2 \frac{\log n}{n^{1/8}} \right) \right) \sum_{S \subseteq V_y | |S| = k} \prod_{v \in S} \Pr(g(v) = y) \\
\leq (1 + o(1)) \frac{1}{k!} \left( \sum_{S \subseteq V_y | |S| = k} \Pr(g(v) = y) \right)^k \\
= (1 + o(1)) \frac{\mathbb{E}(Z^1_y)^k}{k!}.
\]
So (18), (21) imply
\[
\Pr \left( Z^1_y > 2 \frac{n}{\delta^{1/2}} \right) \leq (1 + o(1)) \frac{\mathbb{E}(Z^1_y)^k}{(2n/\delta^{1/2})^k} \leq (1 + o(1))2^{-k}
\]
and then together with (19) we get
\[
\Pr \left( \exists y : Z_y > 2 \frac{n}{\delta^{1/2}} \right) \leq 2n((1 + o(1))2^{-k} + n^{-3}) = o(1). \tag{27}
\]
Case 2: \( y \leq n^{1/4} \).

Assume that \( V_y \neq \emptyset \). We apply the Hoeffding-Azuma inequality. Changing the value of \( g \) on a single edge can only change the value of \( Z^1_y \) by at most 2. Also, \( Z^1_y \) is determined by the outcome of at most
\[
\sum_{v \in V_y} d_v \leq |V_y|(y + (\log n)^2)
\]
random choices. It follows that for $t > 0$,
\[
\Pr(Z_y^1 \geq E(Z_y^1) + t) \leq \exp \left\{ -\frac{t^2}{2|V_y|(y + (\log n)^2)} \right\} .
\] (28)

Putting $t = n/\delta^{1/2}$ and observing that $V_y \neq \emptyset$ implies $\delta \leq n^{1/4}$ and $y\delta \leq n^{1/2}$, and applying (18), (19), (28), we see that
\[
\Pr \left( Z_y^1 > 2 \frac{n}{\delta^{1/2}} \right) \leq e^{-n^{1/2}/3} .
\] (29)

The lemma follows from (19), (27) and (29).

4 Proofs of Theorems

We are now able to prove the Theorems.

**Proof of Theorem 1.** Let $\Delta \leq n^{1/2}$. By Lemma 8, there exists a weight assignment $g : E \to [w]$ with $m_g \leq 6n/\delta$ and $w = 3$. Now by Lemma 4, $s(G) \leq 3m_g w + m_g \leq 60n/\delta$, proving (b). Similar arguments, using Lemma 7 and Lemma 9 in place of Lemma 8, provide part (a) and (c).

**Proof of Theorem 2.** The proof is similar to the proof of Theorem 1, just use Lemma 5 in place of Lemma 4.

**Proof of Theorem 3.** The proof is similar to the proof of Theorem 1, just use Lemma 4 and Lemma 10.

References


