Breaking the $1/(\mu(1-\rho))$ barrier: M/M/1/SRPT has average sojourn time $\Theta((\mu(1-\rho)\log 1/(1-\rho))^{-1})$

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Abstract

We consider an M/M/1 queueing system under the Shortest Remaining Processing Time (SRPT) policy. We show that there are constants c_1 and c_2 such the average sojourn time under SRPT lies between $c_1(\mu(1-\rho)\log 1/(1-\rho))^{-1}$ and $c_2(\mu(1-\rho)\log 1/(1-\rho))^{-1}$, where μ denotes the service rate and ρ denotes the load. Comparing this with the classic result that any scheduling policy that does not use the knowledge of job sizes has average sojourn time $(\mu(1-\rho))^{-1}$, implies that SRPT offers a non-constant improvement over such policies.

1 Introduction

It is a classic result that the average sojourn time in an $M/M/1$ system with First Come First Serve (FCFS) scheduling policy (or any other policy which does not make use of job sizes while scheduling) is $(\mu(1-\rho))^{-1}$ [1]. Here, μ, λ and ρ as usual, refer to the service rate, the arrival rate and the load (λ/μ) respectively. A natural question that arises is "How much more can the sojourn time improve if the knowledge of job sizes is used while scheduling?"

In this note we study the average sojourn time under the Shortest Remaining Processing Time (SRPT) policy. Our main result is that SRPT has average sojourn time $\Theta((\mu(1-\rho)\log(\frac{1}{1-\rho}))^{-1})$. We believe this result is mainly interesting for two reasons. First, since SRPT is the optimal policy with respect to average sojourn time [2], our result places a lower bound on the achievable average sojourn time under any scheduling policy in an M/M/1 system. Second, SRPT and other priority based scheduling policies have received lot of attention recently in practical computer systems, because of their ability to substantially reduce the sojourn times, and hence the quality of service received by the users. However, most of this work involves comparing SRPT with other scheduling policies for specific values of loads using simulation or numerical computation. Our results provide an analytical justification for the empirically observed improvements under SRPT at high loads.

Schrage and Miller first obtained the expression for the expected sojourn time for a job of size x under SRPT in the more general $M/G/1$ queueing system [3]. While these results were obtained as back as 1966, it is somewhat surprising that the asymptotic average sojourn time under SRPT has been not been studied previously. This is perhaps in part due to the complicated nature of the expressions for SRPT: Even for the case of exponential job sizes distribution, the simplest expression for the average sojourn time (to the best of our knowledge) involves a double integral. Our results are obtained by simplifying and bounding the expression for sojourn time of a job in various ways.

2 Preliminaries

We begin with some notation and previous results. Let $E[T(x)]$ denote the expected sojourn time for a job of size x under the SRPT policy, where the job sizes are distributed exponentially with mean μ

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(i.e. $f(t) = \mu e^{-\mu t}$). As usual, the sojourn time of a job is divided in two parts, the waiting time and the residence time. We will use $E[T(x)]$, $E[W(x)]$ and $E[R(x)]$ to denote the expected sojourn time, waiting time, residence time for a job of size x. Based on the result of Schrage and Miller [3] we have,

$$E[W(x)] = \frac{\rho}{\mu} \cdot \frac{(1-(1+\mu x)e^{-\mu x})}{(1-\rho(1-(1+\mu x)e^{-\mu x}))^2} \tag{1}$$

$$E[R(x)] = \int_0^x \frac{dt}{1-\rho(1-(\mu t+1)e^{-\mu t})} \tag{2}$$

Our goal is to analyze $E[T]$: which is clearly given by $E[T] = \int_0^\infty E[T(x)]\mu e^{-\mu x}dx$.

3 Result

Our main result is that

Theorem 1 *For all* $2/3 \leq \rho < 1$,

$$\frac{1}{18e} \cdot \frac{1}{\mu(1-\rho)} \frac{1}{\log(\frac{1}{1-\rho})} \leq E[T] \leq 7 \cdot \frac{1}{\mu(1-\rho)} \frac{1}{\log(\frac{1}{1-\rho})} \tag{3}$$

This result will follow directly from Lemmas 1, 2 and 3 given below. The main idea of the proof is to first simplify the expressions for the residence time and waiting time while losing only a small constant factor, and second to suitably upper and lower bound these expressions.

3.1 Upper Bounding the Sojourn Time

We begin by showing that for large ρ the contribution of the mean residence time to the mean sojourn time is not significant.

Lemma 1

$$E[R] \leq \frac{3}{2\mu} \cdot \log \frac{1}{1-\rho} \tag{4}$$

Proof Sketch: We first upper bound $E[R(x)]$ as follows:

$$E[R(x)] = \int_0^x \frac{dt}{1-\rho(t)} \leq \int_0^x \frac{dt}{1-\rho(x)} = \frac{x}{1-\rho(x)}$$

As, $E[R] = \int_0^\infty E[R(x)]\mu e^{-\mu x}dx \leq \frac{x}{1-\rho(x)}\mu e^{-\mu x}dx$, and observing that $\frac{x}{1-\rho(x)}$ is increasing in $\rho(x)$ and that $\mu e^{-\mu x}dx$ is decreasing in $\rho(x)$. By expressing the above integral is terms of $\rho(x)$ and applying the Chebyshev Integral Inequality to upper bound the integral of the product on an increasing and a decreasing function, we get the required result.

We now consider the waiting time. We will show that

Lemma 2

$$E[W] \leq \frac{11}{2\mu} \cdot \frac{1}{(1-\rho)\log\frac{1}{1-\rho}} \tag{5}$$

Proof Sketch: Simplify Equation 1 we first show that,

$$E[W] \leq \rho \int_0^\infty \frac{e^{-\mu x}}{(1-\rho+\rho(\mu x+1)e^{-\mu x})^2}dx \tag{6}$$

To bound the integral in 6, we split it in two parts at α, where $\alpha = -1/\mu + \frac{1}{3\mu}\log\frac{1}{1-\rho}$, i.e.

$$E[W] \leq \int_0^{\alpha} \frac{\rho e^{-\mu x}}{(1-\rho+\rho(\mu x+1)e^{-\mu x}))^2}dx + \int_{\alpha}^{\infty} \frac{\rho e^{-\mu x}}{(1-\rho+\rho(\mu x+1)e^{-\mu x})^2}dx$$

The final step is to further simplify each of these terms, and upper bound them suitably. We show that the first term is at most $(\mu(1-\rho)\log(\frac{1}{1-\rho}))^{-1}$ and that the second term is at most

$$\frac{9}{2\mu}\frac{1}{(1-\rho)\log\frac{1}{1-\rho}}$$

3.2 Lower bounding the Sojourn Time

We now obtain a matching lower bound (up to constants) on $E[T]$.

Lemma 3

$$E[T] \geq \frac{1}{9\mu}\frac{1}{2e}\frac{1}{(1-\rho)(\log(\frac{1}{1-\rho}))} \tag{7}$$

We proof of this lemma uses ideas similar to those in the previous ones, but is rather long, so we omit the sketch due to lack of space.

4 Numerical Evaluation

In the results above we did not focus on optimizing the constants c_1 and c_2. However, it would be very interesting to see if the gap between c_1 and c_2 could be closed in the asymptotic case.

To get an estimate on the actual value of the constants c_1 and c_2, we evaluate the average sojourn time under M/M/1/SRPT, using the symbolic computation package Mathematica^{TM}. Table below shows the average sojourn time under SRPT for various values of ρ approaching 1. The last column indicates the ratio of $E[T]$ at load ρ and the quantity $\left((1-\rho)\mu\log\left(\frac{1}{1-\rho}\right)\right)^{-1}$, and hence the range of values where the constants c_1 and c_2 might actually lie. The value of c_1 and c_2 seem to converge around 0.8.

Load (ρ)	$1-\rho$	$E[T]$ with $\mu = 1$	$E[T] \times \mu(1-\rho) \times \log(\frac{1}{1-\rho})$
0.9	0.1	3.5521	0.8179
0.95	0.05	5.5410	0.8299
0.99	0.01	17.6269	0.8117
0.995	0.005	30.3829	0.8048

References

[1] R. W. Conway, W. L. Maxwell, and L. W. Miller. *Theory of Scheduling.* Addison-Wesley Publishing Company, 1967.

[2] L.E. Schrage. A proof of the optimality of the shortest processing remaining time discipline. *Operations Research*, 16:678–690, 1968.

[3] L.E. Schrage and L.W. Miller. The queue M/G/1 with the shortest processing remaining time discipline. *Operations Research*, 14:670–684, 1966.