The cover time of sparse random graphs.

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January 8, 2003

Abstract

We study the cover time of a random walk on graphs $G \in G_{n,p}$ when $p = \frac{c \log n}{n}, c > 1$. We prove that \textbf{w.h.p} the cover time is asymptotic to $c \log \left(\frac{e}{c-1}\right) n \log n$.

1 Introduction

Let $G = (V, E)$ be a connected graph, let $|V| = n$, and $|E| = m$. For $v \in V$ let $C_v$ be the expected time taken for a simple random walk $W$ on $G$ starting at $v$, to visit every vertex of $G$. The \textit{cover time} $C_G$ of $G$ is defined as $C_G = \max_{v \in V} C_v$. The cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [1] that $C_G \leq 2m(n-1)$. It is also known (see Feige [6], [7]), that for any connected graph $G$

$$(1 - o(1))n \log n \leq C_G \leq (1 + o(1))\frac{4}{27}n^3.$$  

In this paper we study the cover time of the random graph, $G \in G_{n,p}$. It was shown by Jonasson [10] that \textbf{w.h.p}

\begin{itemize}
  \item[(a)] $C_G = (1 + o(1))n \log n$ if $\frac{np}{\log n} \rightarrow \infty$.
  \item[(b)] If $c > 1$ is constant and $np = c \log n$ then $C_G > (1 + \alpha)n \log n$ for some constant $\alpha = \alpha(c)$.
\end{itemize}

Thus Jonasson has shown that when the expected average degree $(n - 1)p$ grows faster than $\log n$, a random graph has the same cover time \textbf{w.h.p} as the complete graph $K_n$, whose cover

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time is determined by the Coupon Collector problem. Whereas, when \( np = \Omega(\log n) \) this is
not the case.

In this paper we sharpen Jonasson’s results for the case \( np = c \log n \) where \( \omega = (c - 1) \log n \to \infty \). This condition on \( \omega \) ensures that whp \( G_{n, p} \) is connected, (see Erdős and Rényi [5]).

**Theorem 1.** Suppose that \( np = c \log n = \log n + \omega \) where \( \omega = (c - 1) \log n \to \infty \) and \( c = O(1) \).

If \( G \in G_{n, p} \), then whp

\[
C_G \sim c \log \left( \frac{c}{c - 1} \right) n \log n.
\]

In the next section we give some properties that hold whp in \( G_{n, p} \). In Section 3 we show that a graph with these properties has a cover time described by Theorem 1.

## 2 Properties of \( G_{n, p} \)

Let \( \delta, \Delta \) denote the minimum and maximum degree, and let \( d(u, v) \) denote the distance between the vertices \( u, v \) of the graph \( G \).

Let \( np = c \log n \) where \( c > 1 \). Whp \( G \in G_{n, p} \) has the structural properties P0–P7 given below. We say that a graph \( G \) with these properties is *typical*. The proof of the following lemma is given in the Appendix.

**Lemma 1.** Let \( p = \frac{\log n}{n} \) where \( \omega = (c - 1) \log n \to \infty \) and \( c = O(1) \). Then whp \( G \in G_{n, p} \) is typical.

**P0:** \( G \) is connected.

**P1:** \( \Delta(G) \leq \Delta_0 = (c + 10) \log n \) and

\[
\delta(G) \geq \begin{cases} 
1 & c \leq 1 + e^{-500} \\
\alpha \log n & c > 1 + e^{-500} 
\end{cases}
\]

where \( \alpha = \alpha^* / 2 \) and \( \alpha^* > e^{-600} \) satisfies \( c - 1 = \alpha^* \log(ce / \alpha^*) \).

**P2:** There are at most \( n^{1/3} \) small vertices (i.e. of degree at most \( \log n / 20 \)) and no two small vertices are within distance \( \leq \frac{\log n}{(\log \log n)^3} \) of each other.

**P3:** For \( L \subseteq V, |L| \leq 4 \), let \( H = G - L \). For \( S \subseteq V - L \) let \( e_h(S, \overline{S}) \) be the number of edges of \( H \) with one end in \( S \) and the other in \( \overline{S} = V - (L \cup S) \).

For all \( H \subseteq G \) such that \( \delta(H) \geq 1 \), and for all \( S \subseteq V - L, |S| \leq n/2 \),

\[
\frac{e_h(S, \overline{S})}{d_H(S)} \geq \frac{1}{6}.
\]

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P4: Let $\overline{D}(k) = n{\binom{n-1}{k}}p^k(1-p)^{n-1-k}$ denote the expected size of $D(k)$ in $G_{n,p}$. Let $D(k)$ be the number of vertices of degree $k$ in $G$. Define

$$K_0 = \{k \in [1, \Delta_0] : \overline{D}(k) \leq (\log n)^{-2}\}.$$

$$K_1 = \{1 \leq k \leq 15 : (\log n)^{-2} \leq \overline{D}(k) \leq \log \log n\}.$$

$$K_2 = \{k \in [16, \Delta_0] : (\log n)^{-2} \leq \overline{D}(k) \leq (\log n)^2\}.$$

$$K_3 = [1, \Delta_0] \setminus (K_0 \cup K_1 \cup K_2).$$

P4a: If $k \in K_3$ then $\frac{1}{2}\overline{D}(k) \leq D(k) \leq 2\overline{D}(k)$, and

$$D(k) = \begin{cases} 
0 & k \in K_0 \\
(\log \log n)^2 & k \in K_1 \\
(\log n)^4 & k \in K_2
\end{cases}$$

P4b: If $\omega \geq (\log n)^{2/3}$ then $K_1 = \emptyset$ and

$$\min\{k \in K_2\} \geq (\log n)^{1/2} \quad \text{and} \quad |K_2| = O(\log \log n).$$

P5: The number of edges $m = m(G)$ of $G$ satisfies $|m - \frac{1}{2}cn \log n| \leq n^{1/2} \log n$.

P6: Let $k^* = \lceil (c - 1) \log n \rceil$, $V^* = \{v : d(v) = k^* \}$ and let $B^* = \{v \in V^* : \text{dist}(v, w) \leq \frac{10 \log n}{(\log \log n)^2} \text{ for some } w \in V^*, w \neq v\}$. Then

$$|V^*| \geq \frac{1}{2}\overline{D}(k^*) \quad \text{and} \quad |B^*| \leq \frac{1}{10}\overline{D}(k^*).$$

Let $X = \{v : \delta_v \leq \alpha \log n\}$ where $\delta_v \geq 2$ is the minimum degree of a neighbour of $v$, excluding neighbours of degree one. Then

$$|V^* \cap X| \leq \frac{1}{10}\overline{D}(k^*).$$

P7 The minimum distance between two small cycles of length $\leq \frac{\log n}{10 \log \log n}$ is at least $\frac{\log n}{\log \log n}$ and the minimum distance between a small vertex and a small cycle is at least $\frac{\log n}{10 \log \log n}$.

3 The cover time of a typical graph

In this section $G$ denotes a fixed graph with vertex set $[n]$ which satisfies P0–P7 and $u$ is some arbitrary vertex from which a walk is started. For a subgraph $H$ of $G$ let $W_{u,H}$ denote a random walk on $H$ which starts at vertex $u$ and let $W_{u,H}(t)$ denote the walk generated by the first $t$ steps. Let $X_{u,H}(t)$ be the vertex reached at step $t$ and let $P_{u,H}^{(t)}(v) = \Pr(X_{u,H}(t) = v)$. Let $\pi_{u,H}(v)$ be the steady state probability of the random walk $W_{u,H}$. For an unbiased random
walk on a connected graph $H$ with $m(H)$ edges, $\pi_H(v) = \pi_{u,H}(v) = \frac{d_H(v)}{2m(H)}$ where $d_H(v)$ denotes degree in $H$.

Our definition of typical does not rule out $G$ being bipartite, even though $G_{n,p}$ is non-bipartite whp for these values of $p$. In which case there is no steady state distribution. We therefore assume that in such a case, at each step, the random walk does nothing with probability $1/2$ and only moves to an adjacent vertex with probability $1/2$. We double the expected time to cover the vertices, but the asymptotic number of non-trivial steps remains the same.

Let $H(v) = G - \{v\}$ if $v$ is not a neighbour of a vertex $w$ of degree 1, and let $H(v) = G - \{v, w\}$ if $v$ has a neighbour $w$ of degree 1. (Note that P2 rules out a neighbour having two neighbours of degree 1). For a subgraph $H$ let $N_H(v)$ be the neighbourhood of $v$ in $H$ (i.e. $N_H(v) = N_G(v) \cap V(H)$). When $H = G$ we drop the $H$ from the above notation and often drop the $u$ as well.

**Lemma 2.** Let $G$ be typical, then there exists a sufficiently large constant $K > 0$ such that if $\tau_0 = K\log n$ then for all $v \in V$, and for all $u, x \in H = H(v)$, after $t \geq \tau_0$ steps

$$|P_{u,H}^{(t)}(x) - \pi_{u,H}(x)| = O(n^{-10}).$$  

(1)

**Proof** The conductance $\Phi$ of the walk $W_{u,H}$ is defined by

$$\Phi(W_{u,H}) = \min_{\pi(S) \leq 1/2} \frac{e_H(S : \overline{S})}{d_H(S)}.$$

It follows from P3 that the conductance $\Phi$ of the walk $W_{u,H}$ satisfies $\Phi \geq \frac{1}{4}$. Now it follows from Jerrum and Sinclair [9] that

$$|P_{u,H}^{(t)}(x) - \pi_{u,H}(x)| = O(n^{1/2} \left(1 - \frac{\Phi^2}{2}\right)^t).$$  

(2)

For sufficiently large $K$, the RHS above will be $O(n^{-10})$ at $\tau_0$. We remark that there is a technical point here. The result of [9] assumes that the walk is lazy, and only makes a move to a neighbour with probability $1/2$ at any step. This halves the conductance but still (2) remains true. For us it is sufficient simply to keep the walk lazy for $2\tau_0$ steps until it is mixed. This is negligible compared to the cover time.

For $v \neq u \in V$, let $A_t(v)$ be the event that $W_{u,G}(t)$ does not visit $v$.

**Lemma 3.**

(a) If $t \geq 2\tau_0$ and $\delta_v \geq 2$ then

$$\Pr(A_t(v)) \leq \left(1 - \left(\frac{\delta_v - 1}{\delta_v}\right)^2 - O\left(\frac{1}{\log n}\right)\right) \frac{d(v)}{2m} \Pr(A_{2m}(v))$$

$$\Pr(A_t(v)) \geq \left(1 - \left(\frac{\delta_v}{\delta_v - 1}\right)^2 + O\left(\frac{1}{\log n}\right)\right) \frac{d(v)}{2m} \Pr(A_{2m}(v))$$
(b) Suppose that \( v, v' \in V^* \setminus X \) (see P6) and that \( \text{dist}(v, v') > \frac{10 \log n}{(\log \log n)^{1/2}} \). Then

\[
\Pr(\mathcal{A}_{2m}(v) \cap \mathcal{A}_{2m}(v')) \leq \left( 1 - \left( 1 + O\left( \frac{1}{\log n} \right) \right) \right)^{t - 2\tau_0}.
\]

**Proof**

(a) Fix \( w \neq v \) and \( y \in N_H(v) \). Let \( \mathcal{W}_k(y) \) denote the set of walks in \( H(v) \) which start at \( w \), finish at \( y \), are of length \( 2\tau_0 \) and which leave a vertex in the neighbourhood \( N_H(v) \) exactly \( k \) times. (Note that the walk can leave \( y \in N_H(v) \) without necessarily leaving \( N_H(v) \)). Let \( \mathcal{W}_k = \bigcup_y \mathcal{W}_k(y) \) and let \( W = (w_0, w_1, \ldots, w_{2m}) \in \mathcal{W}_k(y) \). Let

\[
\rho_W = \frac{\Pr(X_{w,G}(s) = w_s, s = 0, 1, \ldots, 2\tau_0)}{\Pr(X_{w,H}(s) = w_s, s = 0, 1, \ldots, 2\tau_0)}.
\]

Then

\[
1 \geq \rho_W \geq \left( \frac{\delta_v - 1}{\delta_v} \right)^k.
\]

This is because

\[
\frac{\Pr(X_{w,H}(s) = w_s \mid X_{w,H}(s - 1) = w_{s-1})}{\Pr(X_{w,G}(s) = w_s \mid X_{w,G}(s - 1) = w_{s-1})} = \begin{cases} 
1 & w_{s-1} \notin N_G(v) \\
\frac{d_{G}(w_{s-1})}{d_{G}(w_{s-1}) - 1} & w_{s-1} \in N_G(v)
\end{cases}
\]

If \( \mathcal{E} = \{X_{w,G}(\tau) \neq v, 0 \leq \tau \leq 2\tau_0\} \) then

\[
\Pr(\mathcal{E}) = \sum_{k \geq 0} \sum_{w \in \mathcal{W}_k} \Pr(W_{w,G}(2\tau_0) = W)
\]

\[
= \sum_{k \geq 0} \sum_{w \in \mathcal{W}_k} \rho_W \Pr(W_{w,H}(2\tau_0) = W)
\]

\[
\geq \sum_{k \geq 0} p_k \left( \frac{\delta_v - 1}{\delta_v} \right)^k
\]

where

\[
p_k = \sum_{W \in \mathcal{W}_k} \Pr(W_{w,H}(2\tau_0) = W) = \Pr(W_{w,H}(2\tau_0) \in \mathcal{W}_k).
\]

We will show later that

\[
p_0 + p_1 + p_2 \geq 1 - O((\log n)^{-1})
\]

which immediately implies that

\[
\Pr(\mathcal{E}) \geq p_0 + p_1 \left( 1 - \frac{1}{\delta_v} \right) + p_2 \left( 1 - \frac{1}{\delta_v} \right)^2 \geq \left( 1 - \frac{1}{\delta_v} \right)^2 - O((\log n)^{-1}).
\]
Now fix \( y \) and write
\[
\Pr(X_{w,G}(2\tau_0) = y \mid \mathcal{E}) = \sum_{k \geq 0} \Pr_{W \in \mathcal{W}_k(y)}(W \mid \mathcal{E})^{-1}
\]
\[
= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k(y)} \rho_W \Pr(W \mid \mathcal{E})^{-1}
\]

Now if
\[
p_{k,y} = \frac{\Pr(W_{w,H} \in \mathcal{W}_k(y))}{\Pr(X_{w,H}(2\tau_0) = y)}
\]
\[
= \Pr(W_{w,H}(2\tau_0) \text{ leaves a vertex of } N_H(v) \text{ } k \text{ times} \mid X_{w,H}(2\tau_0) = y)
\]
then
\[
\sum_{k \geq 0} p_{k,y} \left( \frac{\delta_v - 1}{\delta_v} \right)^k \leq \frac{\Pr(X_{w,G}(2\tau_0) = y \mid \mathcal{E})}{\Pr(X_{w,H}(2\tau_0) = y)} \leq \Pr(\mathcal{E})^{-1}.
\]

We will show later that
\[
p_{0,y} + p_{1,y} + p_{2,y} \geq 1 - O((\log n)^{-1})
\]
and so
\[
\left( \frac{\delta_v - 1}{\delta_v} \right)^2 - O\left( \frac{1}{\log n} \right) \leq \left| \frac{\Pr(X_{w,G}(2\tau_0) = y \mid \mathcal{E})}{\Pr(X_{w,H}(2\tau_0) = y)} \right| \leq \left( \frac{\delta_v}{\delta_v - 1} \right)^2 + O\left( \frac{1}{\log n} \right).
\]

Taking \( w \) as \( X_{u,G}(t - 2\tau_0 - 1) \), and conditioning on \( A_{t-2m-1}(v) \), we deduce that
\[
\left( \frac{\delta_v - 1}{\delta_v} \right)^2 - O\left( \frac{1}{\log n} \right) \leq \left| \frac{\Pr(X_{w,G}(t-1) = y \mid A_{t-1}(v))}{\Pr(X_{w,H}(2\tau_0) = y)} \right| \leq \left( \frac{\delta_v}{\delta_v - 1} \right)^2 + O\left( \frac{1}{\log n} \right).
\]

Therefore
\[
\Pr(A_t(v) \mid A_{t-1}(v)) \geq 1 - \left( \frac{\delta_v}{\delta_v - 1} \right)^2 + O\left( \frac{1}{\log n} \right) \sum_{y \in N_H(v)} P_{w,H(v)}^{(2m)}(y) \frac{1}{d(y)}
\]
\[
= 1 - \left( \frac{\delta_v}{\delta_v - 1} \right)^2 + O\left( \frac{1}{\log n} \right) \sum_{y \in N_H(v)} \left( \frac{d(y) - 1}{2m - 2d(v)} + O\left( \frac{1}{n^{10}} \right) \right) \frac{1}{d(y)}
\]
\[
\geq 1 - \left( \frac{\delta_v}{\delta_v - 1} \right)^2 + O\left( \frac{1}{\log n} \right) \left( \frac{d(v)}{2m} - \frac{1}{2m - 2d(v)} \sum_{y \in N_H(v)} \frac{1}{d(y)} \right)
\]
\[
= 1 - \left( \frac{\delta_v}{\delta_v - 1} \right)^2 + O\left( \frac{1}{\log n} \right) \frac{d(v)}{2m}.
\]
Here we use $P_2$ to see that $\sum_{y \in N_H(v)} \frac{1}{d(y)} \leq \frac{40d(v)}{\log n}$.

Similarly,
$$\Pr(A_{t}(v) \mid A_{t-1}(v)) \leq 1 - \left( \left( \frac{\delta - 1}{\delta} \right)^2 - O \left( \frac{1}{\log n} \right) \right) \frac{d(v)}{2m}$$
and the lemma follows immediately.

**Proof of (4,5).** Clearly, we only need to prove (5) and so fix $y \in N_H(v)$.

Let $W(a,b,t)$ denote the set of walks in $H$ from $a$ to $b$ of length $t$ and for $W \in W(a,b,t)$ let $Pr(W) = Pr(W_{a,H}(t) = W)$. Then for $x \in V(H)$ we have

$$Pr(X_{w,H}(\tau_0) = x \mid X_{w,H}(2\tau_0) = y) = \sum_{W_1 \in W(w,x,\tau_0)} \frac{Pr(W_1)Pr(W_2)}{Pr(W(w,y,2\tau_0))}$$
$$= \pi_{x,H}^{-1} \sum_{W_1 \in W(w,x,\tau_0)} \frac{Pr(W_1)\pi_{x,H}Pr(W_2)}{Pr(W(w,y,2\tau_0))}$$

and with $W_3$ equal to the reversal of $W_2$,

$$= \pi_{x,H}^{-1} \pi_{y,H} \sum_{W_1 \in W(w,x,\tau_0)} \frac{Pr(W_1)Pr(W_3)}{Pr(W(w,y,2\tau_0))}$$
$$= \frac{\pi_{x,H}^{-1} \pi_{y,H}}{Pr(W(w,y,2\tau_0))} \Pr(W(w,x,\tau_0))\Pr(W(y,x,\tau_0))$$
$$= \frac{\pi_{x,H}^{-1} \pi_{y,H}}{Pr(W(w,y,2\tau_0))} (\pi_{x,H} - O(n^{-10}))^2$$
$$= \pi_{x,H} - O(n^{-9} \log n).$$

It follows that the variation distance between $X_{w,H}(\tau_0)$ and a vertex chosen from the steady state distribution $\pi_H$ is $O(n^{-8} \log n)$. Now given $x = X_{w,H}(\tau_0)$, $W_{w,H}(\tau_0)$ is a random walk of length $\tau_0$ from $w$ to $x$ and $W_2 = (x = X_{w,H}(\tau_0), X_{w,H}(\tau_0 + 1), \ldots, y = X_{w,H}(2\tau_0))$ is a random walk of length $\tau_0$ from $x$ to $y$. For $W \in \bigcup_\xi W(\xi, y, \tau_0)$ let $Q(W)$ be the probability that $(y, X_{w,H}(2\tau_0 - 1), \ldots, X_{w,H}(\tau_0)) = W$. Then we have

$$Q(W) = (1 + O(n^{-8} \log n)) \frac{\pi_{x,H}Pr(W_{\text{reversal}})}{Pr(W(x,y,\tau_0))}$$
$$= (1 + O(n^{-8} \log n)) \frac{\pi_{y,H}Pr(W)}{Pr(W(x,y,\tau_0))}$$
$$= (1 + O(n^{-8} \log n)) \frac{\pi_{y,H} \pi_{x,H}Pr(W)}{\pi_{x,H}Pr(W(x,y,\tau_0))}$$
$$= (1 + O(n^{-8} \log n)) \frac{\pi_{x,H} \pi_{y,H}Pr(W)}{\pi_{y,H}Pr(W(y,x,\tau_0))}$$
Thus if \( W = (w_1, w_2, \ldots, w_{\tau_0}) \) then

\[
Q(W \mid X_{w,H}(\tau_0) = w_1) = (1 + O(n^{-8} \log n)) \frac{\Pr(W)}{\Pr(W(y, x, \tau_0))}
\]

and so the distribution of \( W^\text{rever} \) is within variation distance \( O(n^{-8} \log n) \) of that of a random walk of length \( \tau_0 \) from \( y \) to a vertex \( x \) chosen with distribution \( \pi_H \).

Thus the distribution of a random walk of length \( 2\tau_0 \) from \( w \) to \( y \) and that of \( W_1, W_3 \) \( \text{revers} \) is \( O(n^{-8} \log n) \) where \( W_1, W_3 \) are obtained by (i) choosing \( x \) from the steady state distribution and then (ii) choosing a random walk \( W_1 \) from \( w \) to \( x \) and a random walk \( W_3 \) from \( y \) to \( x \). Furthermore, the variation distance between the distribution of \( W_1 \) and a random walk of length \( \tau_0 \) from \( w \) is \( O(n^{-9}) \). Similarly, the variation distance between distribution of \( W_3 \) and a random walk of length \( \tau_0 \) from \( y \) is \( O(n^{-9}) \).

Now consider \( W_1 \) and let \( Z_t \) be the distance of \( X_{w,H}(t) \) from \( v \). We observe from \textbf{P2} and \textbf{P7} that except for at most one value \( \bar{a} \in J = [1, \frac{\log n}{2(\log \log n)^2}] \) we have

\[
\Pr(Z_{t+1} = a + 1 \mid Z_t = a) \geq 1 - \frac{20}{\log n}, \quad a \in I \setminus \bar{a}.
\]

and this will enable us to prove

\[
\Pr(W_1 \text{ or } W_3 \text{ make a return to } N_H(v)) = O(1/ \log n) \tag{6}
\]

and this implies (5). (Note that a move from \( N_H(v) \) to \( N_H(v) \) has to be counted as a return here.)

To prove (6), let \( t_0 \) be the first time that \( W_1 \) visits \( N_H(v) \). We have to estimate the probability that \( W_1 \) returns to \( N_H(v) \) later on and so we can assume w.l.o.g. that \( w \in N_H(v) \) i.e. \( Z_0 = 1 \).

It follows from \textbf{P2} and \textbf{P7} that

\[
\Pr(Z_i = i + 1, \ i = 1, \ldots, 6 \mid Z_0 = 1) \geq \left( 1 - \frac{40}{\log n} \right)^6. \tag{7}
\]

To check this consider two possibilities:

(a) There is no small vertex in the \( \leq 7 \) neighbourhood \( N_7 \) of \( v \). Since there is at most one edge joining two vertices in \( N_7 \), we see that \( \Pr(Z_{i+1} > Z_i) = 1 - \frac{40}{\log n} \) for \( i = 1, \ldots, 6 \)

and (7) follows.

(b) On the other hand, if there is a small vertex \( x \) in \( N_7 \) then with probability \( \geq 1 - \frac{40}{\log n} \), the first move from \( w \) takes us further away from \( x \) and (7) follows as before.

If \( Z_3 = 4 \) and there is a return to \( N_H(v) \) then there exists \( \tau \leq \tau_0 \) such that \( Z_\tau = 4, Z_{\tau+1} = 3 \) and \( Z_{\tau+2} \leq 3 \). If there is no small vertex within distance 4 of \( v \) then \textbf{P2} and \textbf{P7} imply

\[
\Pr(\exists \tau \leq \tau_0 : Z_\tau = 4, Z_{\tau+1} = 3, Z_{\tau+2} \leq 3) = O \left( \frac{\tau_0}{(\log n)^2} \right). \tag{8}
\]
If there is a unique small vertex within distance 4 of \( v \) and \( Z_6 = 7 \) and there is a return to
\( N_H(v) \) then there exists \( \tau \leq \tau_0 \) such that \( Z_\tau = 7, Z_{\tau+1} = 6 \) and \( Z_{\tau+2} = 5 \) (no small cycles
close to \( v \) now). We can then argue as in (8) that the probability of this \( O\left( \frac{n}{(\log n)^r} \right) \). This
completes the proof of part (a) of the lemma.

(b) We simply run through the proof as in (a), replacing \( v \) by \( v, v' \): \( H = H(v, v') = G - \{v, v'\}, \)
\( N_H(v, v') = N_G(v) \cup N_G(v') \). The proof of (5) remains valid because \( v, v' \) are far apart. \( \square \)

3.1 The upper bound on cover time

From here on, \( A_1, A_2, \ldots \) are a sequence of unspecified positive constants.

Let \( t_0 = \lceil 2m \log \frac{m}{n} \rceil \). We now prove for typical graphs, that for any vertex \( u \in V \)
\[
C_u \leq t_0 + o(m). \tag{9}
\]

Let \( T_G(u) \) be the time taken to visit every vertex of \( G \) by the random walk \( W_u \). Let \( U_t \) be the
number of vertices of \( G \) which have not been visited by \( W_u \) at step \( t \). We note the following:
\[
\Pr(T_G(u) > t) = \Pr(U_t > 0) \leq \min\{1, \mathbb{E} U_t\}, \tag{10}
\]
\[
C_u = \mathbb{E} T_G(u) = \sum_{t>0} \Pr(T_G(u) > t) \tag{11}
\]

It follows from (10,11) that for all \( t \)
\[
C_u \leq t + \sum_{s>t} \mathbb{E} U_s = t + \sum_{v \in V} \sum_{s>t} \Pr(A_s(v)). \tag{12}
\]

Now, by Lemma 3, for \( s > 2\tau_0 \),
\[
\Pr(A_s(v)) \leq \left(1 - \left(\frac{\delta_v - 1}{\delta_v} \right)^2 - \frac{A_1}{\log n} \right) \frac{d(v)}{2m} \right) s^{-2\tau_0} \Pr(A_{2\tau_0}(v))
\]
\[
\leq \exp\left(- \frac{sd(v)}{2m} \left(1 - \frac{A_2}{\log n} \right) \right), \quad \text{if } \delta_v \geq \alpha \log n
\]

where \( \alpha \) is as in \( \textbf{P1} \).

Then from \( \textbf{P4} \),
\[
\mathbb{E} U_s \leq T_3(s) + T_1(s) + T_2(s) + T_X(s) \tag{13}
\]

where
\[
T_3(s) = 2 \sum_{k=1}^{n-1} n \left( \frac{n-1}{k} \right) p^k (1 - p)^{n-1-k} e^{-\frac{d(v)}{2m} \left(1 - \frac{A_2}{\log n} \right) s},
\]
\[ T_i(s) = \sum_{k \in K_i} D(k) e^{-\frac{sk}{2m}(1 - \frac{A^2}{\log n})}, \quad i = 1, 2 \]

and
\[
T_X(s) = \sum_{v \in X} \left( 1 - \left( \frac{\delta_v - 1}{\delta_v} \right)^2 - O \left( \frac{1}{\log n} \right) \frac{d(v)}{2m} \right)^{s - 2\gamma_0} 
\leq 2 \sum_{v \in X} \exp \left\{ - \left( \frac{\delta_v - 1}{\delta_v} \right)^2 - \frac{A^2}{\log n} \frac{sd(v)}{2m} \right\}. 
\]

Now for \( \gamma > 0 \),
\[
\sum_{s = 0}^{\infty} e^{-\gamma s} \leq \gamma^{-1} e^{-\gamma_0}. \tag{14}
\]

Let \( \lambda = \frac{\ln(n)}{2m} \left( 1 - \frac{\ln(n)}{\log n} \right). \) Applying (14) we get
\[
\sum_{s = 0}^{\infty} T_3(s) \leq 3m \sum_{k=1}^{n-1} \frac{n}{k} \binom{n-1}{k} p^k (1-p)^{n-k-1} e^{-k \lambda} 
\leq \frac{6 m}{p} e^\lambda \sum_{k=1}^{n-1} \binom{n}{k+1} p^{k+1} (1-p)^{n-k-1} e^{-(k+1) \lambda} 
\leq \frac{7 m}{p} c \left( 1 - \frac{p + pe^{-\lambda}}{c} \right)^n 
\leq \frac{7 mn}{(c - 1) \log n} e^{-np + npe^{-\lambda}} 
\leq \frac{8 mc^2A^2}{(c - 1) \log n} 
= \ o(m). \tag{15}
\]

We have used the estimation,
\[
np e^{-\lambda} \leq (c \log n) \left( \frac{c - 1}{c} \right) \left( 1 + \frac{1}{c - 1} \right)^{A^2/\log n} \leq (1 + O(n^{-1}))((c - 1) \log n) \left( 1 + \frac{2A^2}{(c - 1) \log n} \right).
\]

Note that we have used \( (c - 1) \log n \to \infty \) to get the second line.

Continuing we get
\[
\sum_{s = 0}^{\infty} T_1(s) \leq A_4 m \sum_{k \in K_1} \frac{(\log \log n)^2}{k} e^{-k \lambda} 
= \ o(m) \tag{16}
\]
since either (i) $\omega \geq (\log n)^{2/3}$ and $K_1 = \emptyset$ or (ii) $\omega < (\log n)^{2/3}$ and $e^\lambda \geq (1 - o(1))(\log n)^{1/3}$.

$$
\sum_{s=t_0+1}^{\infty} T_2(s) \leq A_5 m \sum_{k \in K_2} \frac{(\log n)^4}{k} e^{-k\lambda} = o(m)
$$

(17)

since either (i) $\omega \geq (\log n)^{2/3}$ and $\min\{k \in K_2\} \geq (\log n)^{1/2}$ and $|K_2| = O(\log \log n)$ or (ii) $\omega < (\log n)^{2/3}$ and $e^\lambda \geq (1 - o(1))(\log n)^{1/3}$.

Note now that $\delta_v \geq 2$ and if $v \in X$ (see P6) then from P2 $d(v) \geq \log n/20$. Thus

$$
\sum_{s=t_0+1}^{\infty} T_X(s) \leq \sum_{s=t_0+1}^{\infty} \sum_{v \in X} \exp\left\{-\frac{sd(v)}{10m}\right\}
\leq \sum_{v \in X} \frac{10m}{d(v)} \exp\left\{-\frac{t_0 d(v)}{10m}\right\}
\leq \sum_{v \in X} \frac{200m}{\log n} \exp\left\{-\frac{t_0 \log n}{200m}\right\} \quad \text{by P2}
\leq \sum_{v \in X} \frac{200m}{\log n} \left(\frac{c - 1}{c}\right)^{\log n/201}
\leq o(m)
$$

(18)

since either (i) $c \geq 1 + e^{-500}$ and $X = \emptyset$ or (ii) $c < 1 + e^{-500}$, in which case we use $(c - 1)/c \leq e^{-500}$.

As $C_G = \max_{v \in V} C_u$, the upper bound on $C_G$ now follows from (9), (13), (15), (16), (17), (18) and (12) with $t = t_0$. \hfill \square

### 3.2 The lower bound on cover time

For any vertex $u$, we can find a set of vertices $S$ such that at time $t_1 = t_0(1 - \epsilon)$, $\epsilon \to 0$, the probability the set $S$ is covered by the walk $W_u$ tends to zero. Hence $T_G(u) > t_1$ whp which implies that $C_G \geq (1 - o(1))t_0$.

We construct $S$ as follows. Let $k^*, V^*, B^*$ be as defined in Property P6.

Let $S^* = V^* \setminus (B^* \cup X)$ and let

$$
\epsilon = \frac{10 \log \log n}{(c - 1) \log c/(c - 1)} = o(1) \text{ and } \delta = \frac{(\log n)^3}{|S^*|}.
$$

Note that

$$
\overline{D}(k^*) = \Omega \left(\frac{n^{(c-1)\ln(c/(c-1))}}{\sqrt{(c-1) \log n}}\right) = \Omega((\log n)^\alpha)
$$

(19)
for any constant $a > 0$. Then **P6** implies that $|S^*| = \Omega((\log n)^a)$ for any constant $a > 0$.

Now for $v, w \neq u$ let $\mathcal{A}_t(v, w)$ be the event that $W$ has not visited $v$ or $w$ by step $t$.

Let $Q \subseteq S^*$ be given by

$$Q = \{ v \in S^* : \Pr(\mathcal{A}_{2\tau_0}(v)) < 1 - \delta, \text{ or } \Pr(\mathcal{A}_{2\tau_0}(v, w)) < (1 - \delta)^2, \text{ for some } w \in S^* \}.$$ 

Now in time $2\tau_0$, $W$ can visit at most $2\tau_0 + 1$ vertices and so

$$\sum_{v \in V} \Pr(\mathcal{A}_{2\tau_0}(v)) \leq 2\tau_0 + 1 \quad \text{and} \quad \sum_{v, w \in V} \Pr(\mathcal{A}_{2\tau_0}(v, w)) \leq \binom{2\tau_0 + 1}{2}.$$ 

Thus

$$|Q| \leq \frac{2\tau_0 + 1}{\delta} + \frac{2\tau_0(2\tau_0 + 1)}{2(1 - (1 - \delta)^2)} = o(|S^*|).$$ 

Therefore, if $S = S^* \setminus Q$,

$$|S| \geq \frac{D(k^*)}{3}.$$ 

Let $S(t)$ denote the subset of $S$ which has not been visited by $W$ by time $t$. Now

$$E |S(t)| \geq \sum_{v \in S} \left( 1 - \left( 1 + \frac{A_6}{\log n} \right) \frac{k^*}{2m} \right)^{t-2\tau_0} \Pr(\mathcal{A}_{2\tau_0}(v)).$$ 

Setting $t = t_1$ we have

$$E |S(t_1)| = \Omega \left( \frac{n^{(c-1)\log c/(c-1)}}{\sqrt{(c-1)\log n}} \exp \left( - \frac{k^*}{2m} t_1 \right) \right)$$

$$= \Omega \left( \frac{n^{(c-1)\log c/(c-1)}}{\sqrt{(c-1)\log n}} \right)$$

$$= \Omega((\log n)^9). \quad (20)$$

Let $Y_{v, t}$ be the indicator for the event that $W_{v}(t)$ has not visited vertex $v$ at time $t$. As $v, w \in S$ are not adjacent, and have no common neighbours, when we delete $v, w$ the total degree of $H(v, w)$ is $2m - 2d(v) - 2d(w)$, and $d(v) = d(w) = k^*$. It follows from Lemma 3(b) that for $v, w \in S$

$$E (Y_{v, t_1} Y_{w, t_1}) \leq \left( 1 - \left( 1 + O \left( \frac{1}{\log n} \right) \right) \frac{k^*}{m} \right)^{t_1-2\tau_0}$$

$$\leq \left( 1 + o(1) \right) E Y_{v, t_1} E Y_{w, t_1}. \quad (21)$$

It follows therefore that

$$\Pr(S(t_1) \neq 0) \geq \frac{\left( E |S(t_1)| \right)^2}{E |S(t_1)|^2} \geq \frac{1}{\frac{E|S_1||S_1-1|}{(E|S(t_1)|)^2} + (E|S(t_1)|)^{-1}} = 1 - o(1)$$

from (20) and (21).

**Acknowledgement** We thank Johan Jonasson for pointing out a significant error in earlier draft.
References


4 Appendix: Typical graph properties

A proof of $P_0, P_1$ can be found in Bollobás [2] or Janson, Łuczak and Ruciński [8].
A proof of $P_2$ can be found in [3].

$P_3$: Case of $1 \leq s = |S| \leq n/(c \log n)$. 

We first prove that \( \text{whp} \quad e_G(S, S) \leq s \log \log n \). Now
\[
\Pr(\exists S : e_G(S, S) \geq s \log \log n) \\
\leq \left( \frac{n}{s} \right)^{\frac{e(n-s)}{s \log \log n}} \leq \left( \frac{ne}{s} \right)^s \left( \frac{spe}{2 \log \log n} \right)^{s \log \log n} \\
\leq \exp \left( -s \left( \log \log n \cdot \log \left( \frac{2n \log \log n}{cse \log n} \right) - \log \frac{ne}{s} \right) \right) \\
= o(n^{-2}).
\]
By property \( \mathbf{P0} \) and the definition of \( H \), both \( G \) and \( H \) contain no isolated vertices and hence \( d(S) > 0 \). We write \( e(S, \overline{S})/d(S) = 1 - 2e(S, S)/d(S) \). Partition \( S \) into sets \( S_1 \) and \( S_2 \), where \( S_1 \) are the vertices of \( S \) of degree at most \((\log n)/10\). Let \( T_1 \) be the neighbour set of \( S_1 \) in \( S_2 \) and let \( T_2 \) be the neighbour set of \( S_1 \) in \( \overline{S} \). By property \( \mathbf{P1} \) the set \( S_1 \) induces no edges, and the neighbours of vertices of \( S_1 \) are distinct. Thus
\[
\frac{2e_H(S, S)}{d_H(S)} \leq \frac{2(|T_1| + |S_2| \log \log n)}{2|T_1| + |T_2| + |S_2|((\log n)/10 - |L|)} \\
\leq \frac{2 + \log \log n}{(\log n)/10} = o(1),
\]
Now use
\[
d_H(S) = e_H(S, \overline{S}) + 2e_H(S, S).
\]
**Case of** \( n/(c \log n) \leq s \leq n/2 \).

The expected value of \( e_H(S, \overline{S}) \) is at least \( \mu = s(n - s - 4)p \). Thus from Chernoff bounds, for fixed \( s \),
\[
\Pr(\exists S : e_H(S, \overline{S}) \leq \mu/2) \leq \left( \frac{n}{s} \right)^{\frac{2(n-s-4)}{n} \log n} \\
\leq \exp \left( -s \left( \frac{c}{18} \log n - \log \frac{ne}{s} \right) \right) \\
= o(n^{-2}).
\]
We note that \( \mathbb{E} d_H(S) = 2\binom{s}{2}p + s(n - s - |L|)p \). Thus
\[
\Pr(\exists S : d_H(S) \geq \frac{3}{2} \mathbb{E} d_H(S)) \leq \left( \frac{n}{s} \right)^{\frac{3}{2} \mathbb{E} d_H(S)} \\
\leq \exp \left( -s \left( \frac{c}{20} \log n - \log \frac{ne}{s} \right) \right) \\
= o(n^{-2}).
\]
Thus
\[
\frac{e_H(S, \overline{S})}{d_H(S)} \geq \frac{1}{2} s(n - s - 4)p \\
\geq \frac{3}{2} (2\binom{s}{2}p + s(n - s)p) \geq \frac{1}{6}.
\]
P4a: First observe that

$$
\Pr(\exists k \in K_0 : D(k) > 0) \leq \sum_{k \in K_0} \mathcal{D}(k) = \frac{|K_0|}{(\log n)^2} = O\left(\frac{1}{\log n}\right).
$$

Then

$$
\Pr(\exists k \in K_1 : D(k) > (\log \log n)^2) \leq \sum_{k \in K_1} \frac{\mathcal{D}(k)}{(\log \log n)^2} = O\left(\frac{1}{\log \log n}\right).
$$

Similarly,

$$
\Pr(\exists k \in K_2 : D(k) > (\log n)^4) \leq \sum_{k \in K_2} \frac{\mathcal{D}(k)}{(\log n)^4} = O\left(\frac{1}{\log n}\right).
$$

A simple calculation gives that for our range of values of $p$

$$
E(D(k)(D(k) - 1)) = \mathcal{D}(k)^2 \left(1 + O\left(\frac{\log n}{n}\right)\right).
$$

Thus

$$
\text{Var} D(k) = \mathcal{D}(k) \left(1 + O\left(\frac{\mathcal{D}(k) \log n}{n}\right)\right).
$$

Applying the Chebychev inequality we see that

$$
\Pr(D(k) \leq \frac{1}{2} \mathcal{D}(k) \text{ or } D(k) \geq 2 \mathcal{D}(k)) \leq \frac{4 \text{Var} D(k)}{\mathcal{D}(k)^2} = \frac{4}{\mathcal{D}(k)} \left(1 + O\left(\frac{\log n}{n}\right)\right).
$$

So, as $|K_3| = O(\log n)$,

$$
\Pr(\exists k \in K_3 : D(k) \leq \frac{1}{2} \mathcal{D}(k) \text{ or } D(k) \geq 2 \mathcal{D}(k)) = O\left(\frac{1}{\log n}\right).
$$

P4b: The sequence $(\mathcal{D}(k), k \geq 0)$ is unimodal and

$$
\frac{\mathcal{D}(k+1)}{\mathcal{D}(k)} \sim \frac{c \log n}{k+1} \quad \text{when } k = O(\log n). \tag{22}
$$

Moreover, for $k \leq \Delta_0$ there is a positive constant $A = A(k)$ such that

$$
\mathcal{D}(k) \sim A n^{1-c} \left(\frac{c \log n}{k}\right)^k \frac{1}{k^{1/2}}. \tag{23}
$$

Suppose first that there exists $k \in K_1 \cup K_2$ such that $k < (\log n)^{1/2}$. It follows from (23) that $c - 1 < (\log n)^{-1/3}$, for if $c - 1 \geq (\log n)^{-1/3}$ and $k < (\log n)^{1/2}$ then $\mathcal{D}(k) = o((\log n)^{-2})$.

Now suppose that $k \in K_2$ implies $k \geq (\log n)^{1/2}$. Observe from (23) that both $\mathcal{D}([c - c^{1/3}] \log n]$) and $\mathcal{D}([c + c^{1/3}] \log n]$) are much greater than $(\log n)^2$. Thus either $k \leq$
\((c - c^{1/3}) \log n \text{ or } k \geq (c + c^{1/3}) \log n\). In either case, we see from iterating (22) that \(|K_2| = O(\log \log n)\).

**P5:** This follows immediately from Chernoff bounds.

**P6:** From (19) we see that \([(c - 1) \log n] \in K_3\) for \(c\) constant. That \(|V^*| \geq \frac{1}{2} \mathcal{D}(k^*)\) now follows from **P3**. Now \(|B^*| \leq \{(v, w) \in (V^*)^2 : \text{dist}(v, w) \leq d = \frac{10 \log n}{(\log \log n)^2}\}\). Therefore

\[
E |B^*| \leq \mathcal{D}(k^*)^2 \sum_{k=1}^d n^k p^{k+1} = o(\mathcal{D}(k^*)),
\]

and the second part of **P6** follows from the Markov inequality. The third part is a similar first moment calculation.

**P7:** A proof of similar results can be found in [3]. \qed