Covering Graphs using Trees and Stars

Amitabh Sinha

GSIA, Carnegie Mellon University

(Work done while visiting MPII Saarbrucken, Germany)

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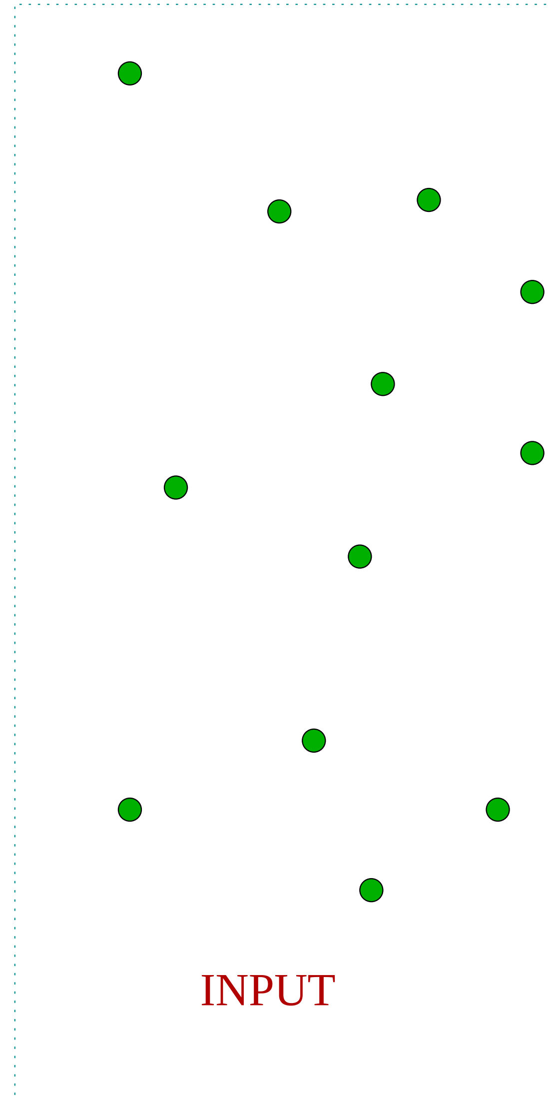
- At 8 am, each nurse begins her "morning round" of patients under her care.
- Morning round ends when *all* nurses have returned to their bases.
- Objective: Assign patients to nurses so that morning rounds end ASAP.

Problem definition

- Input: Graph $G = (V, E)$, edge weights w, integer k.



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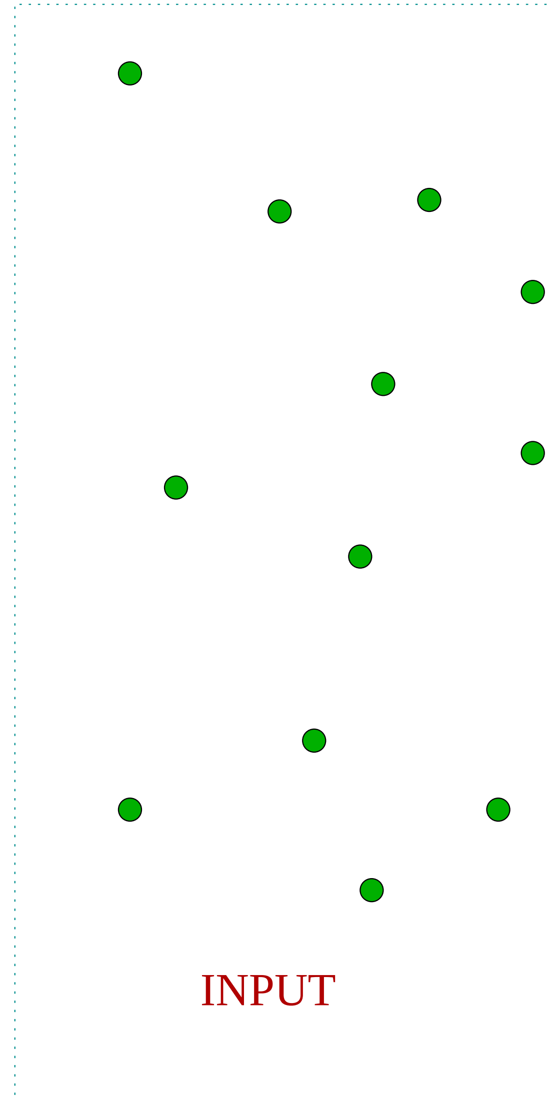

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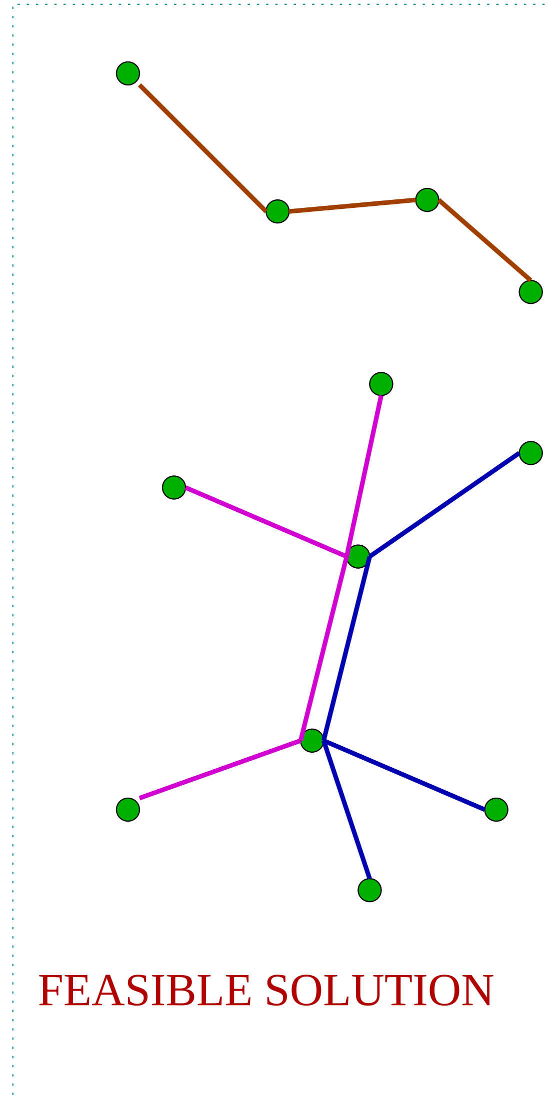

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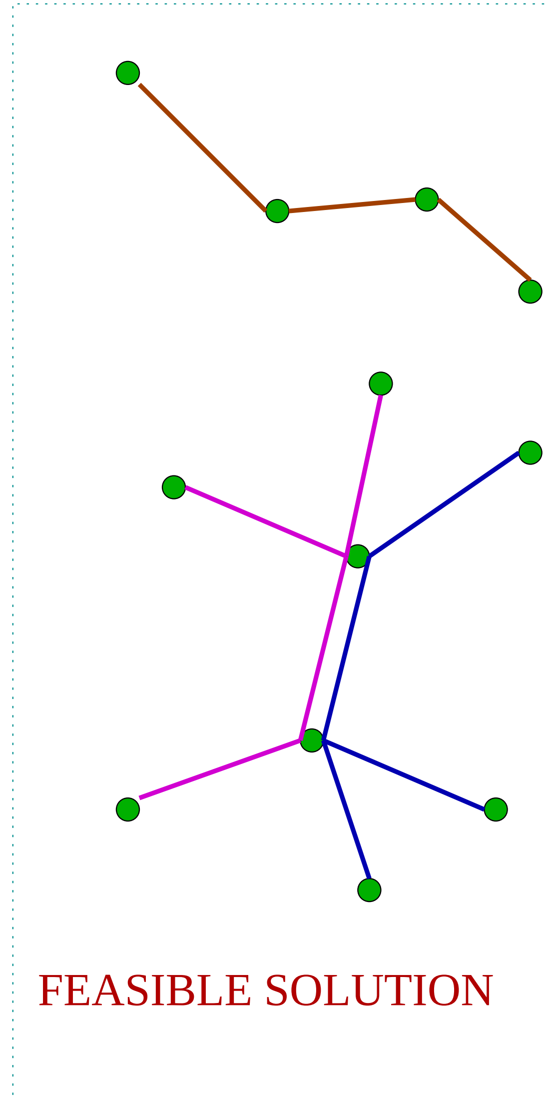

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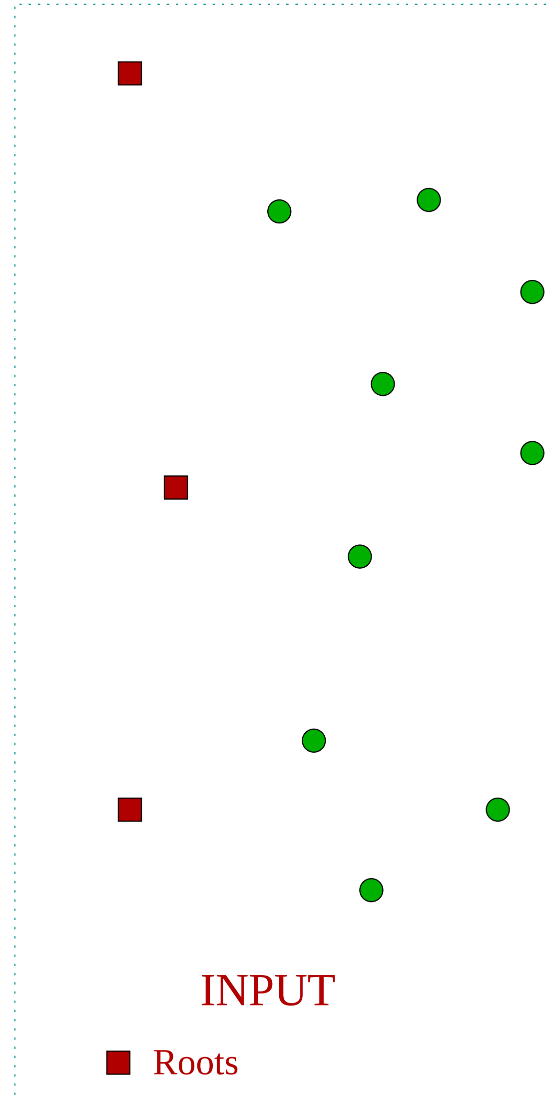

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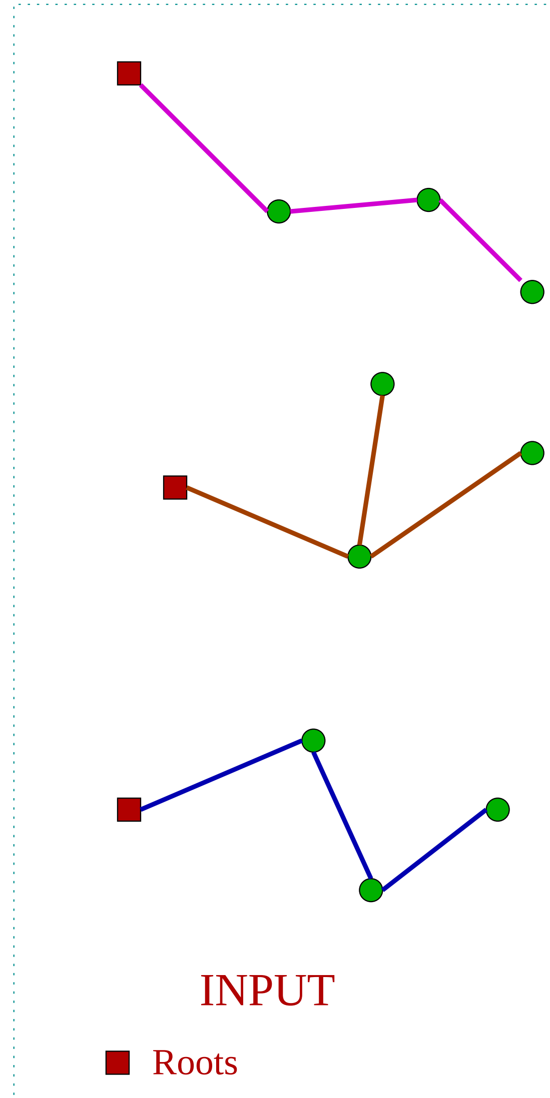

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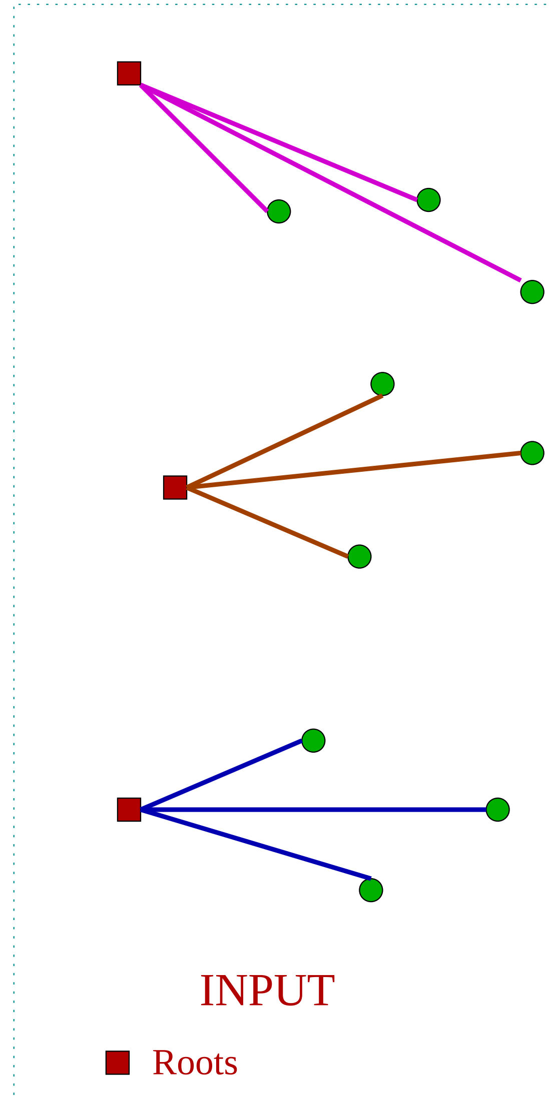
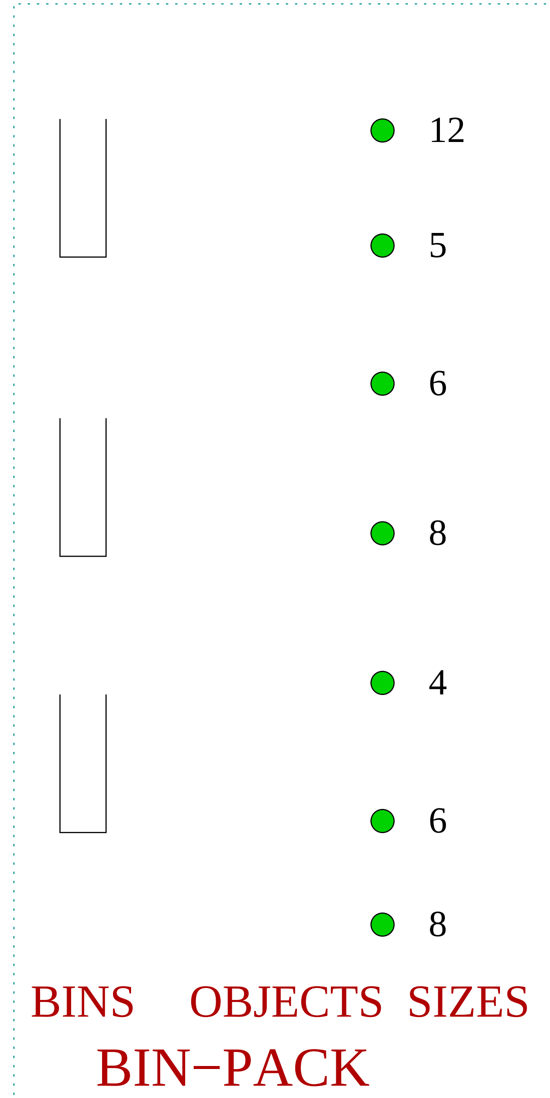
- **Star cover**: Cover with stars, same objective; may be rooted or unrooted.



Hardness (of rooted k-star cover)

- Reduction from BIN-PACK:
  Given elements U with sizes s_u, k bins of size B. Can we pack elements in k bins?



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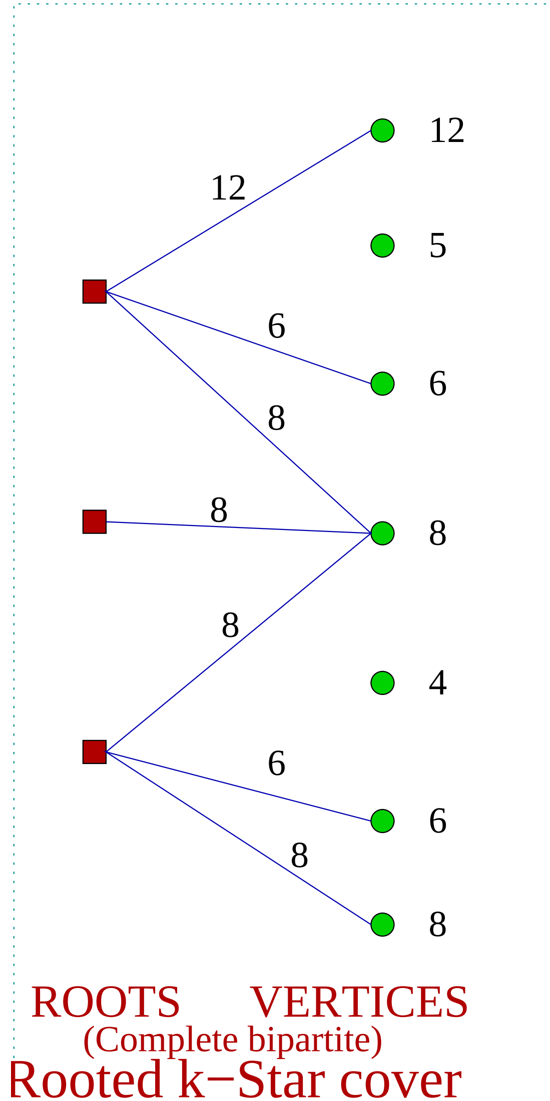

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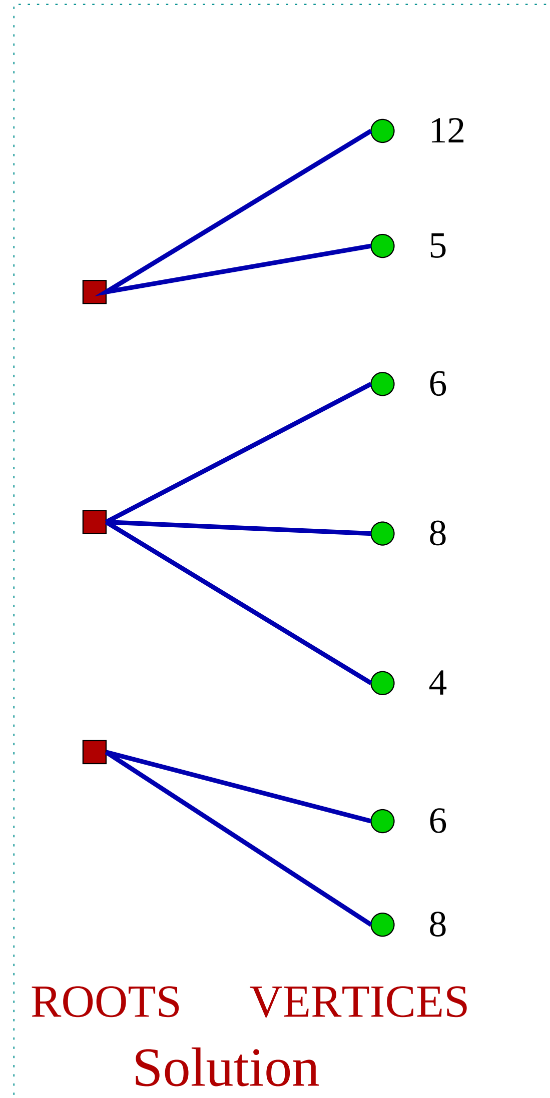
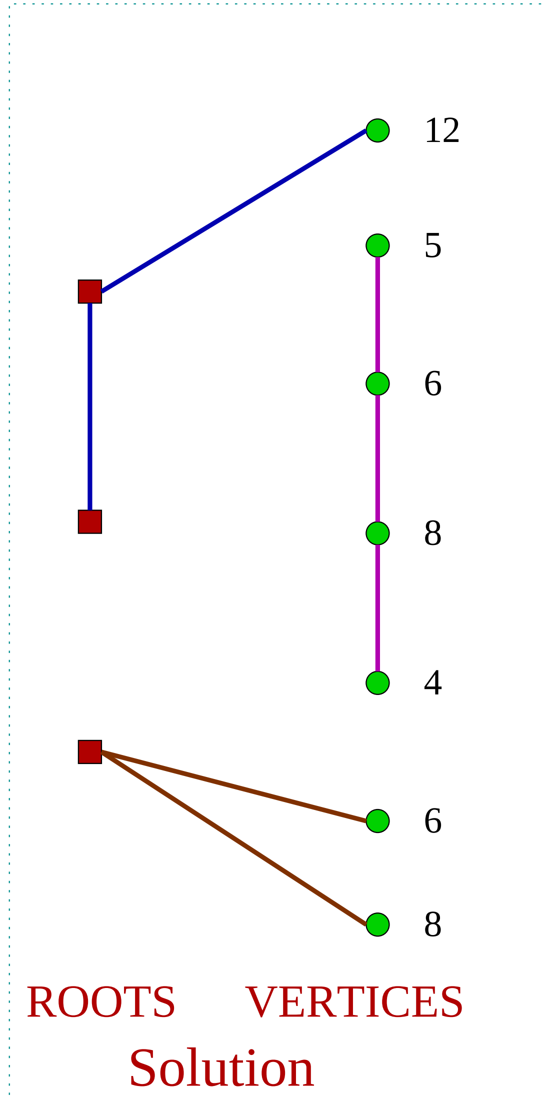
- Claim: BIN-PACK is identical to this special case of Rooted k-star cover.



Hardness of others

- Also by reduction from BIN-PACK.

- Key: Poly time algorithm to convert any solution to Rooted k-star cover solution without increasing cost. (In BIN-PACK graph.)



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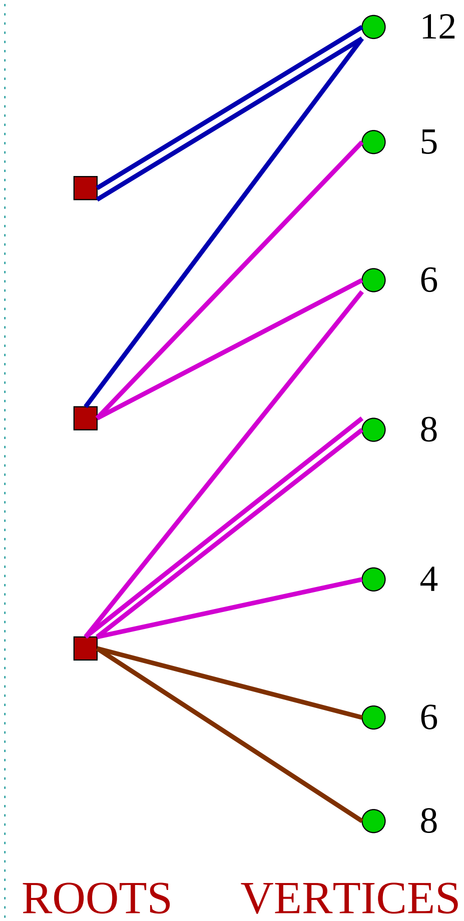

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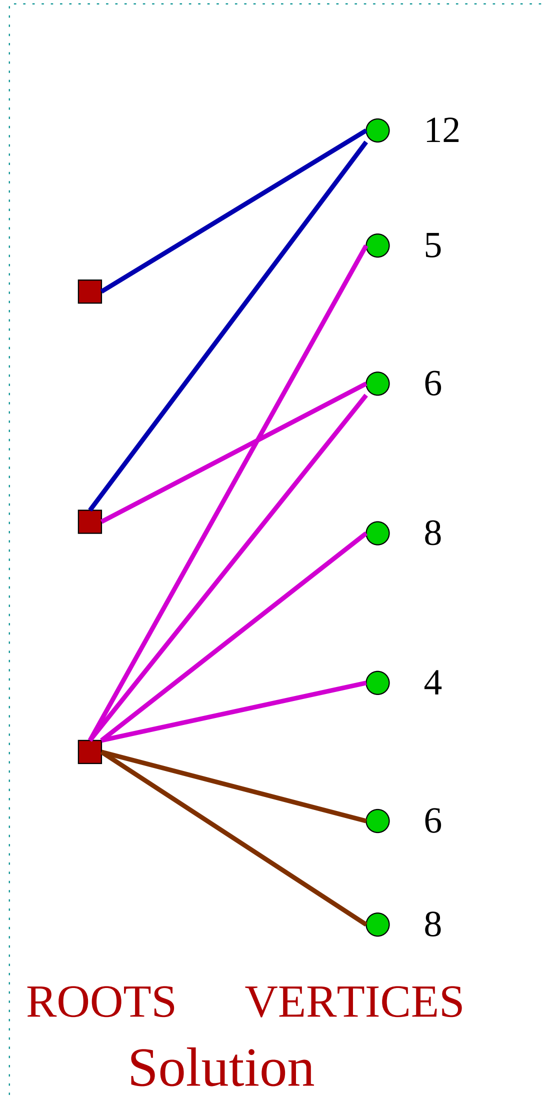

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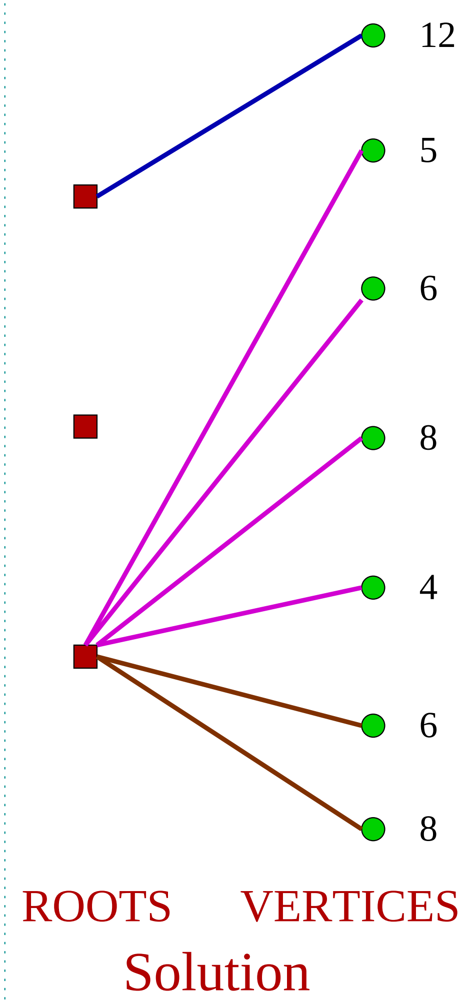

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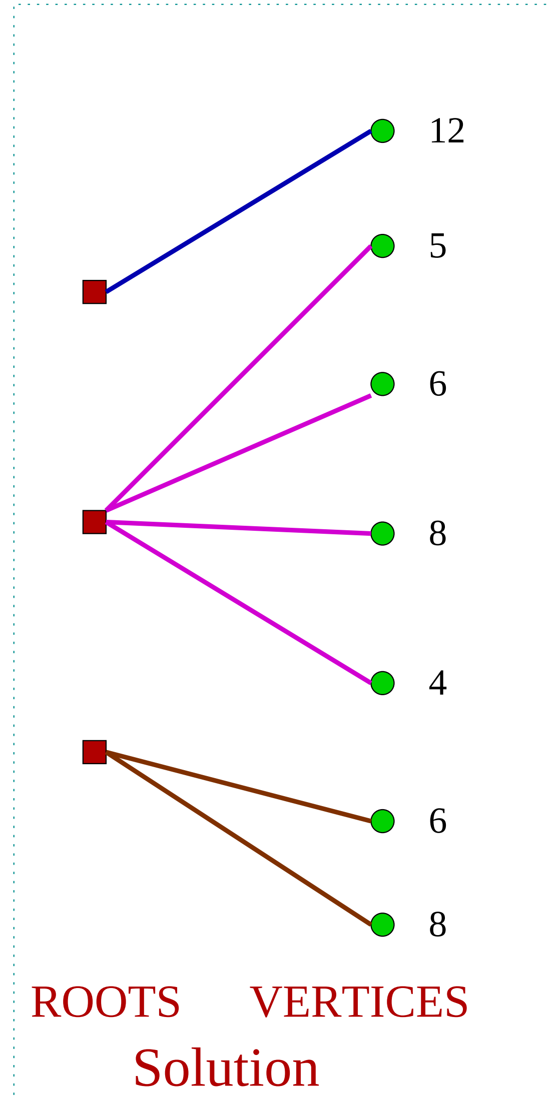

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- Can be made strongly polynomial; approximation ratio worsens to $4 + \epsilon$.

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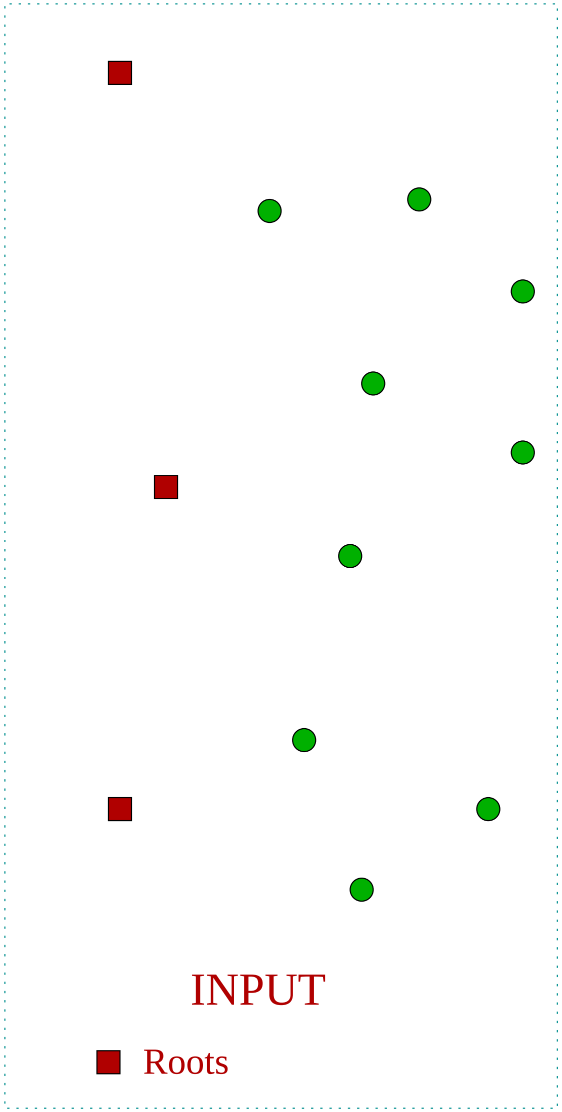
   $w(S_i^j) \in [B, 2B)$ and $w(L_i) < B$.
4. Match trees $\{S_i^j\}_i^j$ to roots in R within distance B from it.
   - If possible, return "success".
   - If impossible, return "fail".

Algorithm: Demonstration

1. Prune.



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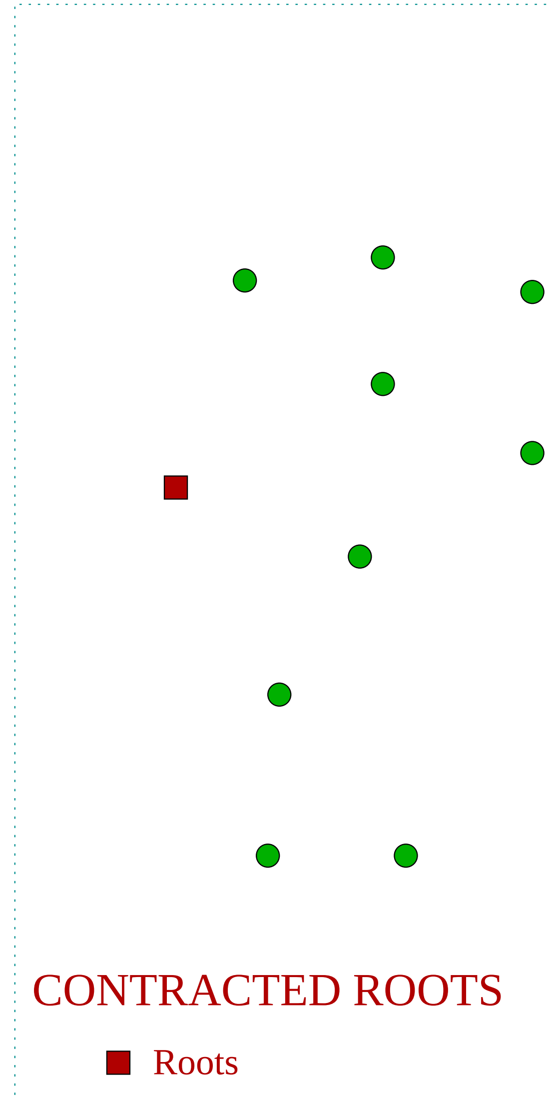

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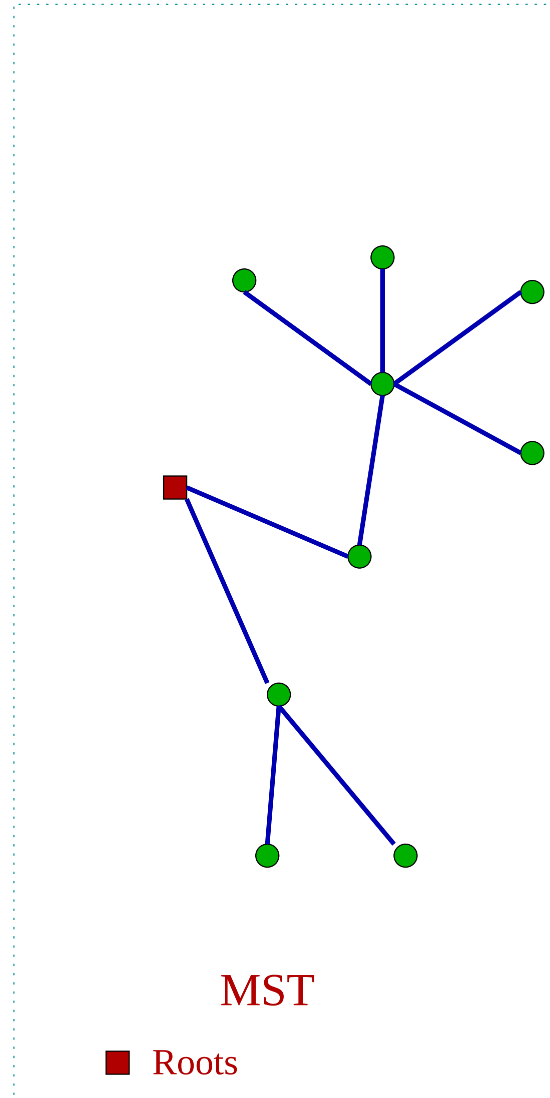

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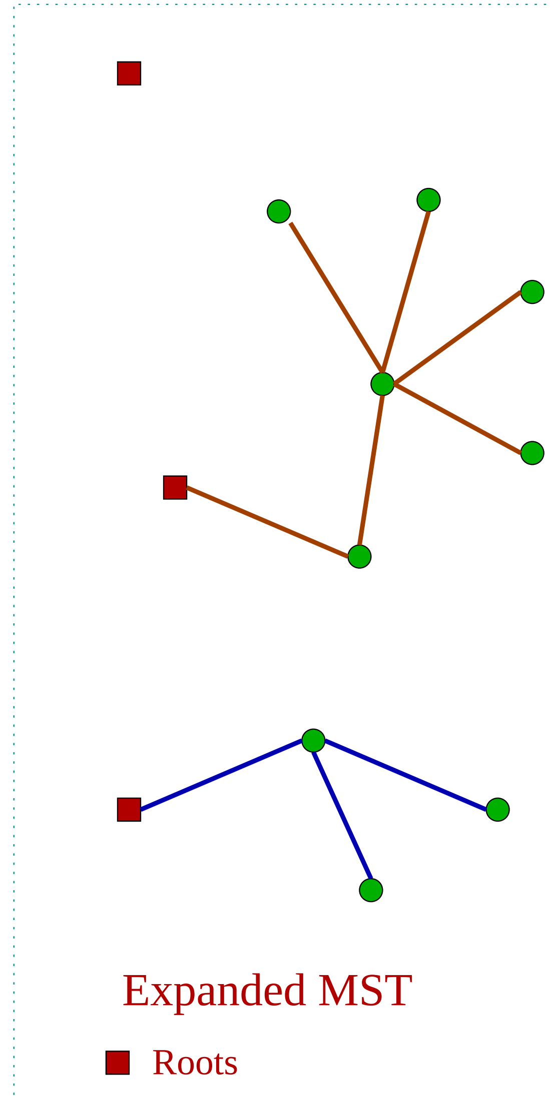

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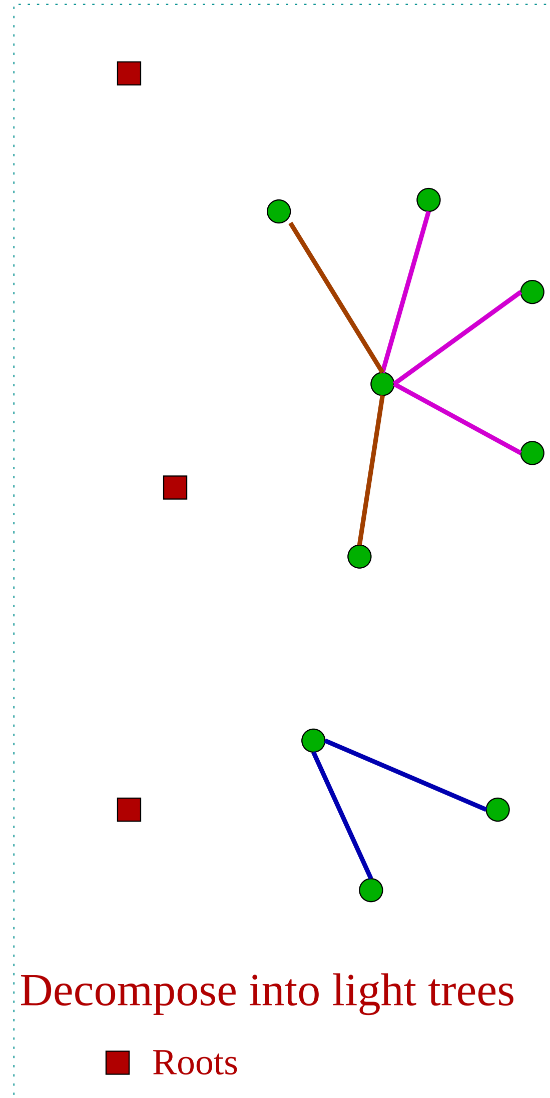

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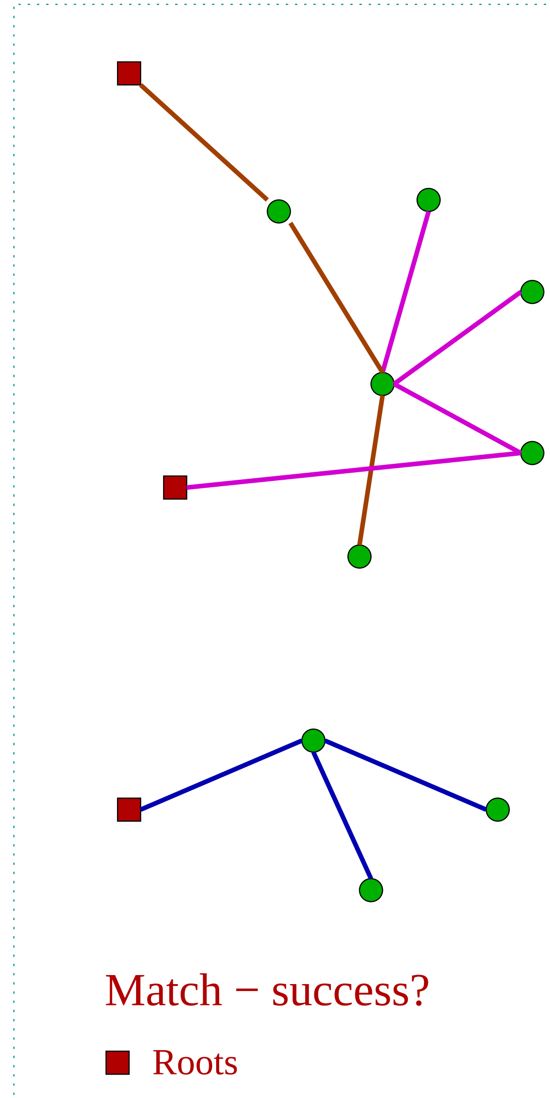

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$B^*|N(S)| \geq B^*|T^*(S)| \geq w(T^*(S)) \geq w(S) \geq B|S|$. $\square$

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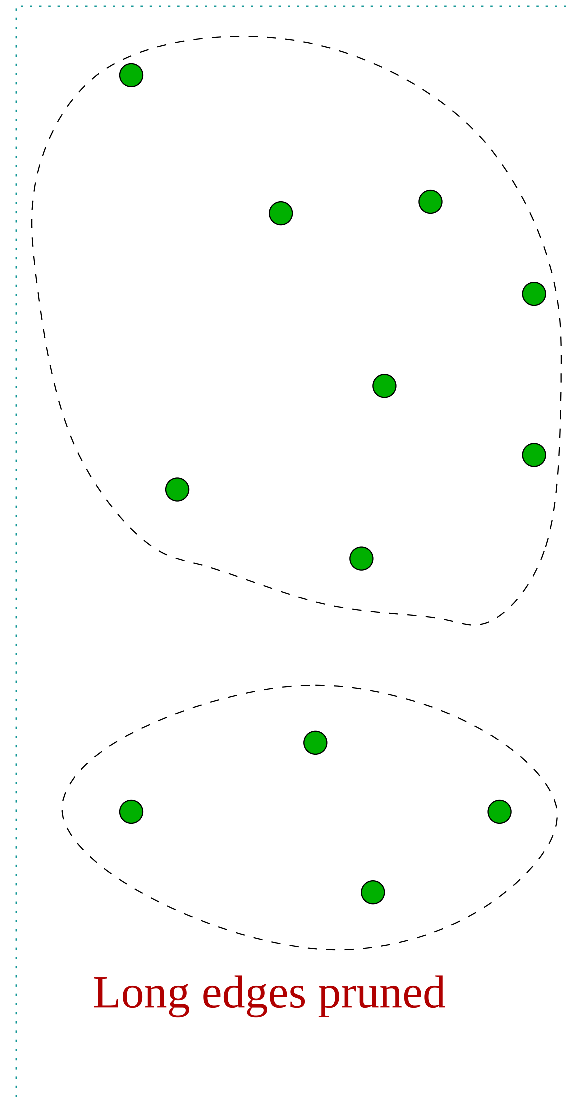

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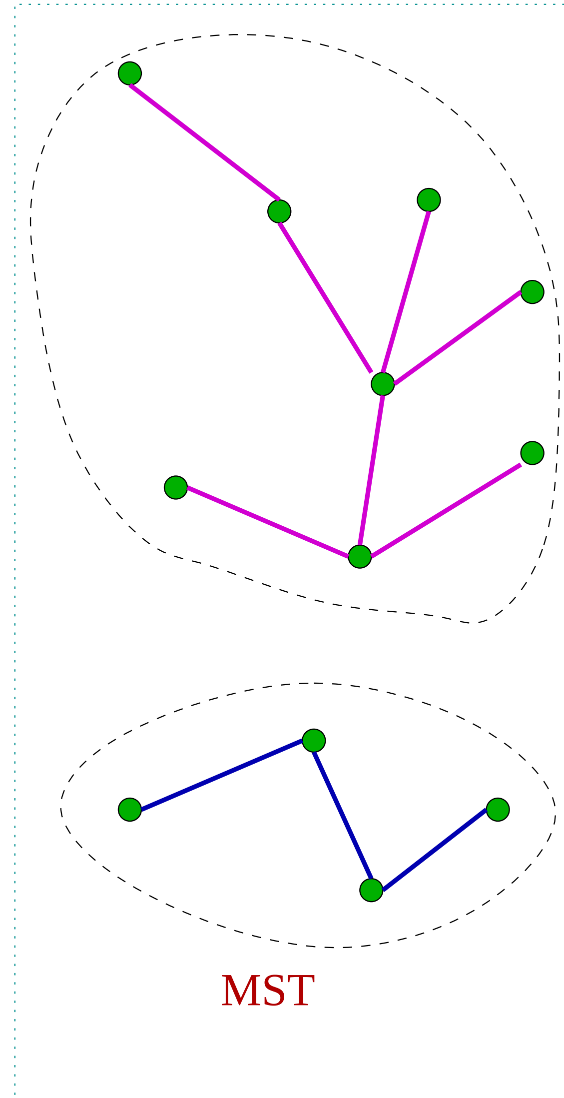

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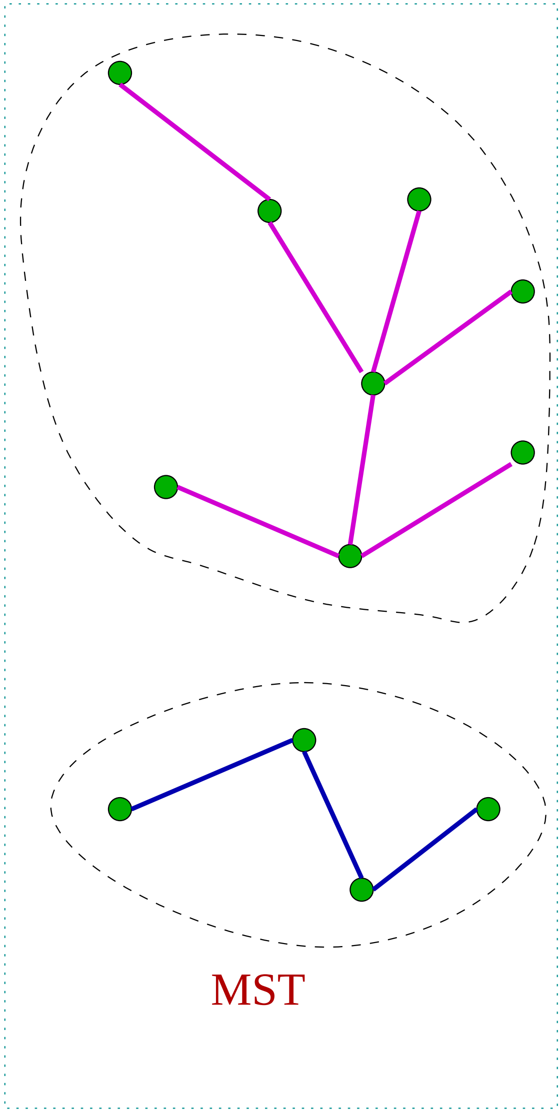

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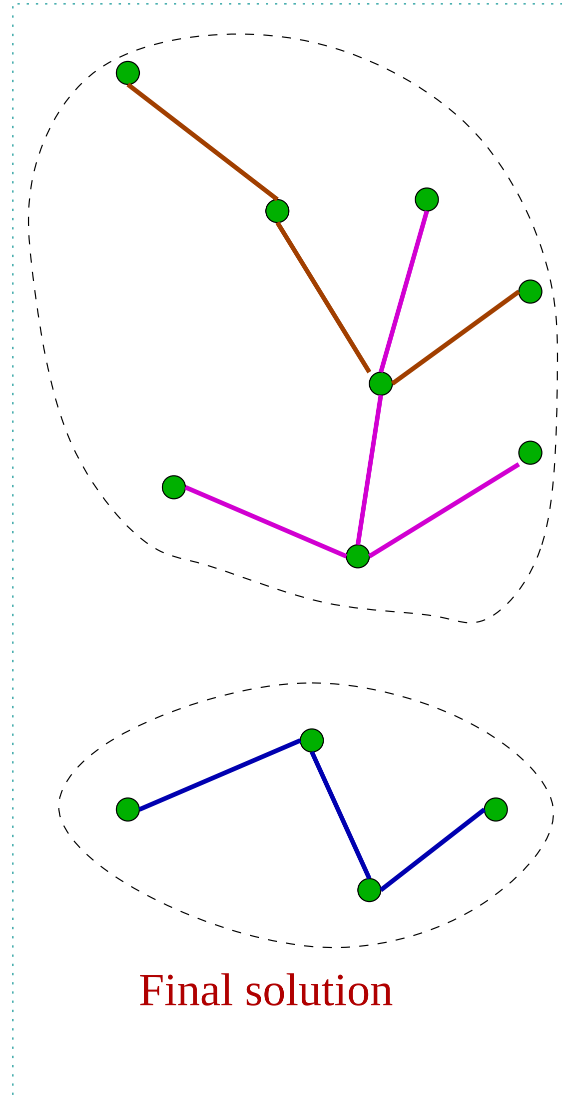

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Therefore $k_i^* \geq \frac{w(MST_i)}{2B} + \frac{1}{2} > k_i$. □

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- Questions?

This research was sponsored in part by National Science Foundation (NSF) grant no. CCR-0122581.