# Solving Partial Differential Equations Numerically

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# Overview

- What are partial differential equations?
- How do we solve them? (Example)
- Numerical integration
- Movie

# Partial Differential Equations

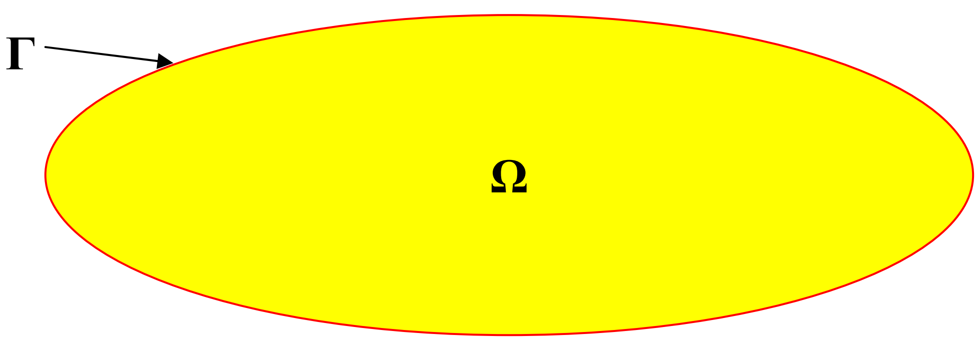
- Equation involving functions and their partial derivatives
- Example: Wave Equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

- We wish to know $\psi$, which is function of many variables
- Typically, no analytical solution possible

# Problem Domain

- Want to solve problem for specific domain
- $\Omega$: bounded open domain in space $R^n$
- $\Gamma$: boundary of $\Omega$
- If domain 2-D, we have following:



# Navier-Stokes Equation

- Model of incompressible fluid flow
- Governed by equation:

$$\boxed{\begin{array}{ll}
\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} = -\dfrac{\nabla P}{\rho} + \mathbf{a} & \text{in } \Omega \times (0, \mathrm{T}] \\
\nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times (0, \mathrm{T}] \\
\mathbf{u} = 0 & \text{on } \Gamma \times (0, \mathrm{T}] \\
\mathbf{u}(\mathbf{x}) = \mathbf{u}^0 & \text{at } t = 0
\end{array}}$$

**u**: fluid velocity

P: pressure

ρ: mass density

ν: dynamic viscosity

**a**: acceleration due to external force

# Simplifying Navier-Stokes: Just Stokes

- Assume steady flow:

$$\boxed{\begin{array}{ll} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\nabla^2\mathbf{u} = -\dfrac{\nabla P}{\rho} + \mathbf{a} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \Gamma \end{array}}$$

- Neglect convection:

$$\boxed{\begin{array}{ll} -\nu\nabla^2\mathbf{u} = -\dfrac{\nabla P}{\rho} + \mathbf{a} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \Gamma \end{array}}$$

# Solving Stokes Equation

$$\boxed{\begin{array}{ll} -\nu\nabla^2\mathbf{u}=\mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u}=0 & \text{in } \Omega \\ \mathbf{u}=0 & \text{on } \Gamma \end{array}}$$

- Space of possible functions $V$ that could be solutions to velocity $\mathbf{u}$
- Choose any $\mathbf{v}\in V$, multiply both sides of Stokes Equation and integrate, resulting in:

$$\boxed{-\nu\int_\Omega \mathbf{v}\cdot\nabla^2\mathbf{u}\,d\mathbf{x}=\int_\Omega \mathbf{v}\cdot\mathbf{f}\,d\mathbf{x}}$$

# Solving Stokes Equation 2

- Apply Green's Theorem and boundary conditions to obtain:

$$\boxed{\nu \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}}$$

- For notational convenience, this is written as:

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v})}$$

# Discretize the domain

- Create mesh by partitioning domain into finite elements
- Create subspace $V_h$ of $V$ of piecewise polynomial functions with basis function defined at each node

$$\boxed{\varphi_i(\mathbf{x}_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}}$$

$$\boxed{\mathbf{v}_h(\mathbf{x}) \in V_h \Rightarrow \mathbf{v}_h(\mathbf{x}) = \sum_{i=1}^{M} \eta_i \varphi_i(\mathbf{x})}$$

# Linear Equations

- Since $V_h \subset V$, $\mathbf{u}_h \in V$ so we can say that in order for $\mathbf{u}_h$ to be a solution to Stokes equation, we need

$$\boxed{\langle \mathbf{u}_h, \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in V_h}$$

- In particular, must be true for basis functions, so writing $\mathbf{u}_h$ in terms of basis functions, we have:

$$\boxed{\sum_{i=1}^{M} \eta_i \langle \varphi_i, \varphi_j \rangle = (\mathbf{f}, \varphi_j) \qquad j = 1,\dots,M}$$

# Solution to Problem

$$\mathbf{A}\boldsymbol{\eta} = \mathbf{b}$$

$$\mathbf{A}_{ij} = \langle \varphi_i, \varphi_j \rangle$$

$$\boldsymbol{\eta} = [\eta_1, \ldots, \eta_M]^T$$

$$\mathbf{b} = [(\mathbf{f}, \varphi_1), \ldots, (\mathbf{f}, \varphi_M)]^T$$

- Knowing coefficients $\eta_i$ means we know solution $\mathbf{u}_h$
- This is a system of M linear equations with M unknowns, can be solved with various numerical methods

# Integration

- Computing stiffness matrix and load vector requires lots of integrals –
  $\langle \varphi_i, \varphi_j \rangle, \quad (\mathbf{f}, \varphi_i)$
- These are done numerically via quadratures:

$$\boxed{\begin{array}{l} \int_0^1 f(x)dx = \sum_{i=1}^{n} w_i f(x_i) \\ \int_0^1 \int_0^{1-y} f(x,y)dxdy = \sum_{i=1}^{n} w_i f\left(x_i, y_i\right) \end{array}}$$

- Choose Gauss points and weights to give high order accuracy

# What I'm working on

- Making this integration fast

Any Questions?