The Directed Deletion Model

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1 The Recurrence

Let $D_{i,j}(t)$ be the random variable measuring the number of vertices of out-degree i and in-degree j in G_t. A rough recurrence for $\overline{D}_{i,j}(t)$ is as follows:

$$\begin{aligned}
\overline{D}_{i,j}(t+1) &= \overline{D}_{i,j}(t) \\
&+ (\alpha - \alpha_1 + \alpha_1(1-\beta))m\mathbf{E}\left[\frac{(i-1)D_{i-1,j}(t)}{e_t} - \frac{iD_{i,j}(t)}{e_t} - O\left(\frac{\Delta_t^{\text{out}}}{e_t}\right)\middle| e_t > 0\right]\mathbf{Pr}[e_t > 0] \\
&+ (\alpha - \alpha_1 + \alpha_1\beta)m\mathbf{E}\left[\frac{(j-1)D_{i,j-1}(t)}{e_t} - \frac{jD_{i,j}(t)}{e_t} - O\left(\frac{\Delta_t^{\text{in}}}{e_t}\right)\middle| e_t > 0\right]\mathbf{Pr}[e_t > 0] \\
&+ (1-\alpha-\alpha_0)\mathbf{E}\left[\frac{(i+1)D_{i+1,j}(t)}{v_t} + \frac{(j+1)D_{i,j+1}(t)}{v_t} - \frac{(i+j+1)D_{i,j}(t)}{v_t}\middle| v_t > 0\right]\mathbf{Pr}[v_t > 0] \\
&+ \alpha_0 m\mathbf{E}\left[\frac{(i+1)D_{i+1,j}(t)}{e_t} + \frac{(j+1)D_{i,j+1}(t)}{e_t} - \frac{(i+j)D_{i,j}(t)}{e_t} + O\left(\frac{\Delta_t^{\text{in}}}{e_t} + \frac{\Delta_t^{\text{out}}}{e_t}\right)\middle| e_t \geq m\right]\mathbf{Pr}[e_t \geq m]
\end{aligned}$$

The terms after this line come with no warranty whatsoever!

$$\begin{aligned}
&+ \alpha_1 \mathbf{1}_{i+j=m}\left(\binom{m}{i}\beta^i(1-\beta)^{m-i} - \mathbf{E}\left[O\left(\frac{\Delta_t^{\text{out}}}{e_t} + \frac{\Delta_t^{\text{in}}}{e_t}\right)\middle| e_t > 0\right]\right)\mathbf{Pr}[e_t > 0] \\
&+ \alpha_1 \mathbf{1}_{i+j<m}\left(O\left(\mathbf{Pr}(e_t < m)\right) + \mathbf{E}\left[O\left(\frac{\Delta_t^{\text{out}}}{e_t} + \frac{\Delta_t^{\text{in}}}{e_t}\right)\middle| e_t > 0\right]\right) + O(\mathbf{Pr}[e_t = 0])
\end{aligned}$$

We define probabilities as follows:

a. With probability $\alpha - \alpha_1$ we add m random directed edges preferentially. For each edge (u, v), choose u, v independently based on

$$\mathbf{Pr}[u = x_i] = \frac{\mathrm{d_{out}}(x_i, t)}{e_t} \tag{1}$$

and

$$\mathbf{Pr}[v = x_i] = \frac{\mathrm{d_{in}}(x_i, t)}{e_t} \tag{2}$$

b. With probability α_1 we add a new vertex x_t. Each of its m incident edges is chosen to be an out-edge or an in-edge with probability β and $1 - \beta$, respectively, and then attached preferentially, as in (1) and (2).

c. With probability $1 - \alpha - \alpha_0$ we delete a vertex chosen uniformly at random.

d. With probability α_0 we delete m edges chosen uniformly at random.

2 Number of edges in G_t

A recurrence for $\mathbf{E}[e_t]$ is presented:

$$\begin{aligned}
\mathbf{E}[e_{t+1}] &= \mathbf{E}\left[e_t + (\alpha - \alpha_0)m - (1 - \alpha - \alpha_0)\sum_{i,j\geq 0}\frac{(i+j)D_{i,j}(t)}{v_t}\right] \\
&\approx \mathbf{E}[e_t] + (\alpha - \alpha_0)m - \frac{2(1-\alpha-\alpha_0)}{\nu t}\mathbf{E}[e_t] \\
&= \mathbf{E}[e_t]\left(1 - \frac{2(1-\alpha-\alpha_0)}{\nu t}\right) + (\alpha - \alpha_0)m
\end{aligned}$$

Setting

$$\mathbf{E}[e_t] = \eta = \frac{m(\alpha - \alpha_0)\nu}{1 + \alpha_1 - \alpha - \alpha_0}$$

gives us the solution.

3 Solving the Recurrence

Let

$$\begin{aligned}
\overline{d}_{i,j}(t) &= \frac{\overline{D}_{i,j}(t)}{t} \\
\gamma_0 &= m(\alpha - \alpha_1 + \alpha_1(1-\beta)) \\
\gamma_1 &= m(\alpha - \alpha_1 + \alpha_1\beta) \\
\gamma_2 &= 1 - \alpha - \alpha_0 \\
\gamma_3 &= m\alpha_0
\end{aligned}$$

Assuming e_t and v_t are concentrated about their mean, we get

$$\begin{aligned}
\overline{D}_{i,j}(t+1) = \overline{D}_{i,j}(t) &+ \left(\gamma_0\frac{i-1}{\eta}\right)\overline{d}_{i-1,j}(t) + \left(\gamma_1\frac{j-1}{\eta}\right)\overline{d}_{i,j-1}(t) \\
&+ \left(-\gamma_0\frac{i}{\eta} - \gamma_1\frac{j}{\eta} - \gamma_2\frac{i+j+1}{\nu} - \gamma_3\frac{i+j}{\eta}\right)\overline{d}_{i,j}(t) \\
&+ \left(\gamma_2\frac{i+1}{\nu} + \gamma_3\frac{i+1}{\eta}\right)\overline{d}_{i+1,j}(t) + \left(\gamma_2\frac{j+1}{\nu} + \gamma_3\frac{j+1}{\eta}\right)\overline{d}_{i,j+1}(t) \\
&+ \alpha_1\mathbf{1}_{i+j=m}
\end{aligned}$$

Assuming $\overline{d}_{i,j} = \overline{d}_{i,j}(t)$ is constant, we get

$$\begin{aligned}
\overline{d}_{i,j} = &\left(\gamma_0\frac{i-1}{\eta}\right)\overline{d}_{i-1,j} + \left(\gamma_1\frac{j-1}{\eta}\right)\overline{d}_{i,j-1} \\
&+ \left(-\gamma_0\frac{i}{\eta} - \gamma_1\frac{j}{\eta} - \gamma_2\frac{i+j+1}{\nu} - \gamma_3\frac{i+j}{\eta}\right)\overline{d}_{i,j} \\
&+ \left(\gamma_2\frac{i+1}{\nu} + \gamma_3\frac{i+1}{\eta}\right)\overline{d}_{i+1,j} + \left(\gamma_2\frac{j+1}{\nu} + \gamma_3\frac{j+1}{\eta}\right)\overline{d}_{i,j+1} \\
&+ \alpha_1\mathbf{1}_{i+j=m}
\end{aligned}$$

Shifting indices to 0 and rearranging,

$$\begin{aligned}
-\alpha_1 \mathbf{1}_{i+j=m-2} &= \left(\gamma_0 \frac{i}{\eta}\right) \overline{d}_{i,j+1} + \left(\gamma_1 \frac{j}{\eta}\right) \overline{d}_{i+1,j} \\
&+ \left(-\gamma_0 \frac{i+1}{\eta} - \gamma_1 \frac{j+1}{\eta} - \gamma_2 \frac{i+j+3}{\nu} - \gamma_3 \frac{i+j+2}{\eta} - 1\right) \overline{d}_{i+1,j+1} \\
&+ \left(\gamma_2 \frac{i+2}{\nu} + \gamma_3 \frac{i+2}{\eta}\right) \overline{d}_{i+2,j+1} + \left(\gamma_2 \frac{j+2}{\nu} + \gamma_3 \frac{j+2}{\eta}\right) \overline{d}_{i+1,j+2}
\end{aligned}$$

Solving this recurrence is hard. Thus, let us to consider a general (homogenous) recurrence of the form

$$\begin{aligned}
0 &= A_0 i f_{i,j+1} + B_0 j f_{i+1,j} \\
&+ (A_1(i+1) + B_1(j+1) + E_1) f_{i+1,j+1} \\
&+ A_2(i+2) f_{i+2,j+1} + B_2(j+2) f_{i+1,j+2}
\end{aligned}$$

The constant will be taken care of later.

4 The Solution Attempt

We use a two variable version of Laplace's method for solving recurrences. Let

$$f_{i,j} = \int_{a'}^{b'} \int_a^b s^{i-1} t^{j-1} v(s,t)\, ds\, dt$$

where a, b, a', b', v are to be determined. Using integration by parts, we get

$$\begin{aligned}
i \cdot f_{i,j} &= \int_{a'}^{b'} \left[\left[s^i t^{j-1} v(s,t)\right]_{s=a}^{s=b} - \int_a^b s^i t^{j-1} \frac{\partial}{\partial s}(v(s,t))\, ds \right] dt \\
&= \int_{a'}^{b'} \left[s^i t^{j-1} v(s,t)\right]_{s=a}^{s=b} dt - \int_{a'}^{b'} \int_a^b s^i t^{j-1} \frac{\partial}{\partial s}(v(s,t))\, ds\, dt \\
j \cdot f_{i,j} &= \int_a^b \left[\left[s^{i-1} t^j v(s,t)\right]_{t=a'}^{t=b'} - \int_{a'}^{b'} s^{i-1} t^j \frac{\partial}{\partial t}(v(s,t))\, dt \right] ds \\
&= \int_a^b \left[s^{i-1} t^j v(s,t)\right]_{t=a'}^{t=b'} ds - \int_{a'}^{b'} \int_a^b s^{i-1} t^j \frac{\partial}{\partial t}(v(s,t))\, ds\, dt
\end{aligned}$$

Let

$$\begin{aligned}
\phi_0(s,t) &= A_0 + A_1 s + A_2 s^2 \\
\phi_1(s,t) &= B_0 + B_1 t + B_2 t^2
\end{aligned}$$

Putting these into our recurrence, we get

$$\begin{aligned}
0 &= \int_{a'}^{b'} \left[s^i t^j \phi_0(s) \cdot v(s,t)\right]_{s=a}^{s=b} dt + \int_a^b \left[s^i t^j \phi_1(t) \cdot v(s,t)\right]_{t=a'}^{t=b'} ds \\
&- \int_{a'}^{b'} \int_a^b s^i t^j \left(\phi_0(s) \cdot \frac{\partial}{\partial s}(v(s,t)) + \phi_1(t) \cdot \frac{\partial}{\partial t}(v(s,t)) - E_1 \cdot v(s,t)\right) ds\, dt
\end{aligned} \tag{3}$$

We are thus left with solving the partial differential equation

$$0 = -E_1 \cdot v(s,t) + \phi_0(s) \cdot \frac{\partial}{\partial s}(v(s,t)) + \phi_1(t) \cdot \frac{\partial}{\partial t}(v(s,t))$$

Because

$$A_0 = \frac{\gamma_0}{\eta} \qquad A_1 = -\frac{\gamma_0}{\eta} - \frac{\gamma_2}{\nu} - \frac{\gamma_3}{\eta}$$

$$B_0 = \frac{\gamma_1}{\eta} \qquad B_1 = -\frac{\gamma_1}{\eta} - \frac{\gamma_2}{\nu} - \frac{\gamma_3}{\eta}$$

$$A_2 = B_2 = \frac{\gamma_2}{\nu} + \frac{\gamma_3}{\eta} \qquad E_1 = -\frac{\gamma_0}{\eta} - \frac{\gamma_1}{\eta} - \frac{\gamma_2}{\nu} - 1$$

we derive

$$\begin{aligned} A_1 &= -(A_0 + A_2) \\ \phi_0(s) &= A_0 - (A_0 + A_2)s + A_2 s^2 \\ &= A_2(s-1)(s - A_0/A_2) \end{aligned}$$

Similarly,

$$\begin{aligned} B_1 &= -(B_0 + B_2) \\ \phi_1(t) &= B_0 - (B_0 + B_2)t + B_2 t^2 \\ &= B_2(t-1)(t - B_0/B_2) \end{aligned}$$

Conjecturing that $v(s,t) = f(s)g(t)$, we get

$$0 = -E_1 + \frac{\phi_0(s) f'(s)}{f(s)} + \frac{\phi_1(t) g'(t)}{g(t)}$$

Setting

$$k = -E_1 + \frac{\phi_0(s) f'(s)}{f(s)} = -\frac{\phi_1(t) g'(t)}{g(t)}$$

We get

$$\begin{aligned} f(s) &= K_0 \left(\frac{s-1}{s - \frac{A_0}{A_2}} \right)^{\frac{-k-E1}{A_0 - A_2}} \\ g(t) &= K_1 \left(\frac{t-1}{t - \frac{B_0}{B_2}} \right)^{\frac{k}{B_0 - B_2}} \\ v(s,t) &= K \left(\frac{s-1}{s - \frac{A_0}{A_2}} \right)^{\frac{-k-E1}{A_0 - A_2}} \left(\frac{t-1}{t - \frac{B_0}{B_2}} \right)^{\frac{k}{B_0 - B_2}} \end{aligned}$$

Let

$$\sigma = \frac{-k - E1}{A_0 - A_2} \qquad \tau = \frac{k}{B_0 - B_2}$$

$$\kappa = \frac{A_2}{A_0} < 1 \qquad \lambda = \frac{B_2}{B_0} < 1$$

We see that to make the left hand side of (3) go to 0, we choose values $a = a' = 0, b = b' = 1$. Assume for now that $A_0 > A_2, B_0 > B_2$. Let

$$\begin{aligned} u_{i,j} &= K \int_0^1 \int_0^1 s^{i-1} t^{j-1} \left(\frac{s-1}{s - \frac{A_0}{A_2}} \right)^{\sigma} \left(\frac{t-1}{t - \frac{B_0}{B_2}} \right)^{\tau} ds\, dt \\ &= K \int_0^1 \int_0^1 s^{i-1} t^{j-1} \left(\frac{1-s}{1-\kappa s} \right)^{\sigma} \left(\frac{1-t}{1-\lambda t} \right)^{\tau} ds\, dt \end{aligned}$$

$E1 < 0$, so we need to choose $k \in [-\infty, -E1]$. Now with the powers positive, we integrate once:

$$\begin{aligned} u_{i,j} &= K \int_0^1 \int_0^1 s^{i-1} t^{j-1} \left(\frac{1-s}{1-\kappa s} \right)^{\sigma} \left(\frac{1-t}{1-\lambda t} \right)^{\tau} ds\, dt \\ &= K \int_0^1 t^{j-1} \left(\frac{1-t}{1-\lambda t} \right)^{\tau} \int_0^1 s^{i-1} \left(\frac{1-s}{1-\kappa s} \right)^{\sigma} ds\, dt \\ &= K \int_0^1 t^{j-1} \left(\frac{1-t}{1-\lambda t} \right)^{\tau} \int_0^1 s^{i-1} (1-s)^{\beta} \sum_{z=0}^{\infty} \frac{\Gamma(\sigma+z)}{\Gamma(\sigma)\Gamma(z+1)} (\kappa s)^z\, ds\, dt \\ &= K \int_0^1 t^{j-1} \left(\frac{1-t}{1-\lambda t} \right)^{\tau} \sum_{z=0}^{\infty} \frac{\Gamma(\sigma+z)}{\Gamma(\sigma)\Gamma(z+1)} \kappa^z \int_0^1 s^{i+z-1} (1-s)^{\beta}\, ds\, dt \\ &= K \int_0^1 t^{j-1} \left(\frac{1-t}{1-\lambda t} \right)^{\tau} \sum_{z=0}^{\infty} \frac{\Gamma(\sigma+z)}{\Gamma(\sigma)\Gamma(z+1)} \frac{\Gamma(i+z)\Gamma(\sigma+1)}{\Gamma(i+z+\sigma+1)} \kappa^z\, ds\, dt \end{aligned}$$

as $i \to \infty$, we use Stirling's formula on $\Gamma(i+z), \Gamma(i+z+\sigma+1)$ to get

$$\begin{aligned} &= K \int_0^1 t^{j-1} \left(\frac{1-t}{1-\lambda t} \right)^{\tau} \left(1 + O\left(i^{-1}\right)\right) \sigma \sum_{z=0}^{\infty} \kappa^z \frac{\Gamma(z+\sigma)}{\Gamma(z+1)} (i+z+\sigma)^{-1-\sigma}\, ds\, dt \\ &= K \int_0^1 t^{j-1} \left(\frac{1-t}{1-\lambda t} \right)^{\tau} \left(1 + O\left(i^{-1}\right)\right) i^{-1-\sigma}\, dt \end{aligned}$$

analogously for the variable t, we get

$$= K \left(1 + O\left(\min\{i,j\}^{-1}\right)\right) i^{-1-\sigma} j^{-1-\tau}$$

5 Boundary Conditions

We consider the case where $i \geq 1, j = 0$. We prove that $u_{i,0} \neq 0$. We have

$$
\begin{aligned}
&A_0 i u_{i,1} + A_1(i+1)u_{i+1,1} + A_2(i+2)u_{i+2,1} + (B_1 + E_1)u_{i+1,1} + 2B_2 u_{i+1,2} \\
&= \int_0^1 \left[s^i \phi_0(s) \cdot v(s,t)\right]_{s=0}^{s=1} \, dt + B_2 \int_0^1 \left[s^i t^2 \cdot v(s,t)\right]_{t=0}^{t=1} \, ds + \int_0^1 \int_0^1 s^i E1 \cdot v(s,t) - s^i \phi_0(s) \cdot \frac{\partial}{\partial s}(v(s,t)) \\
&\quad + B_1 s^i \cdot v(s,t) - B_2 s^i t^2 \cdot \frac{\partial}{\partial t}(v(s,t)) \, ds \, dt \\
&= B_1 \int_0^1 \left[s^i t \cdot v(s,t)\right]_{t=0}^{t=1} \, ds + \int_0^1 \int_0^1 -ks^i \cdot v(s,t) - B_1 s^i t \cdot \frac{\partial}{\partial t}(v(s,t)) - B_2 s^i t^2 \cdot \frac{\partial}{\partial t}(v(s,t)) \, ds \, dt \\
&= \int_0^1 \int_0^1 -ks^i \cdot v(s,t) - s^i(\phi_1(t) - B_0) \cdot \frac{\partial}{\partial t}(v(s,t)) \, ds \, dt \\
&= \int_0^1 \int_0^1 B_0 s^i \cdot \frac{\partial}{\partial t}(v(s,t)) \, ds \, dt + \int_0^1 \int_0^1 -ks^i \cdot v(s,t) - s^i \phi_1(t) \cdot \frac{\partial}{\partial t}(v(s,t)) \, ds \, dt \\
&= \int_0^1 \left[B_0 s^i \cdot v(s,t)\right]_{t=0}^{t=1} \, ds = \int_0^1 -B_0 s^i \cdot v(s,0) \, ds \\
&< 0
\end{aligned}
$$

A similar argument holds for $i = 0, j \geq 1$, so we also have $u_{0,j} \neq 0$.