Topological Representations for Meshes

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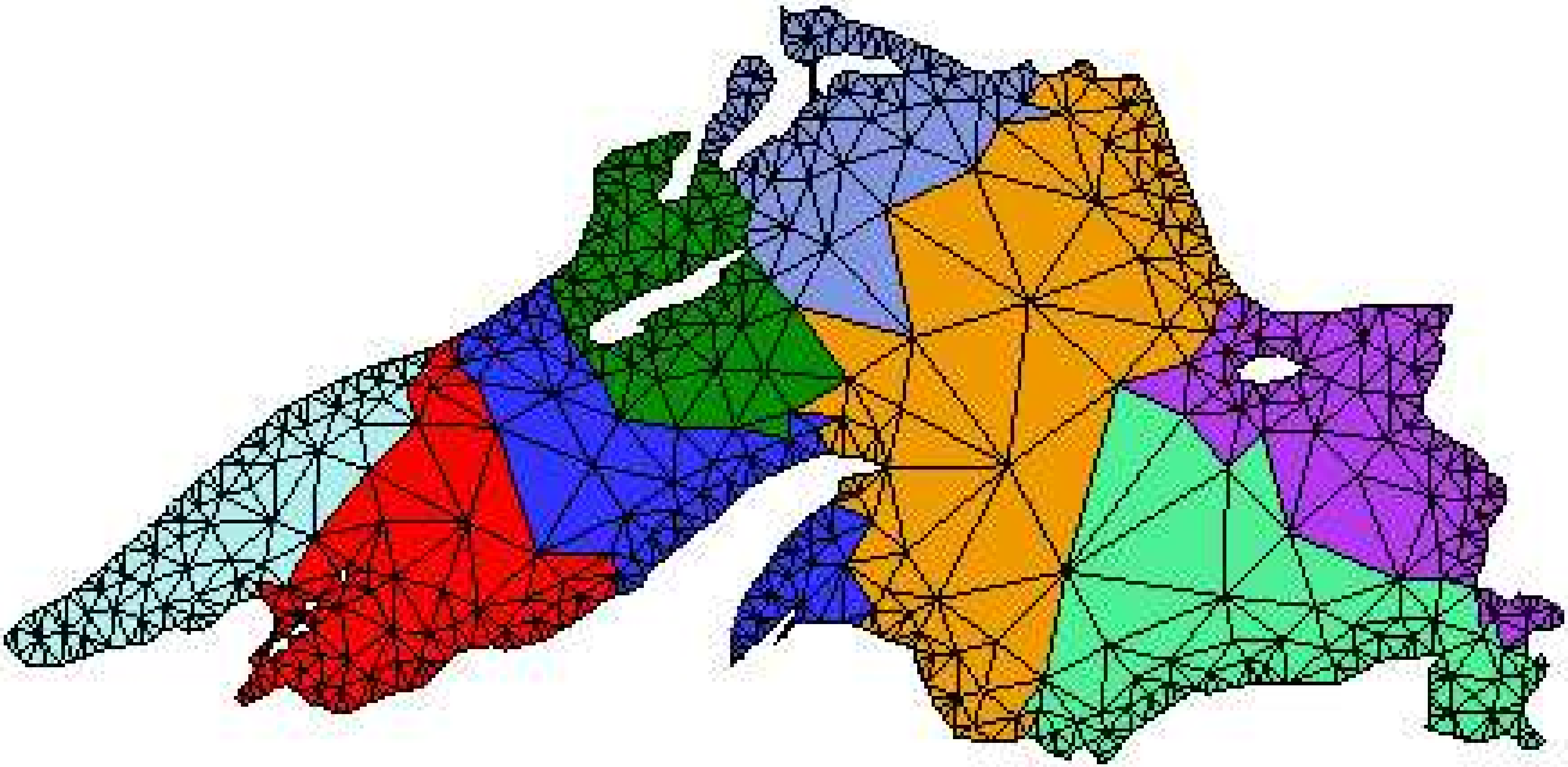

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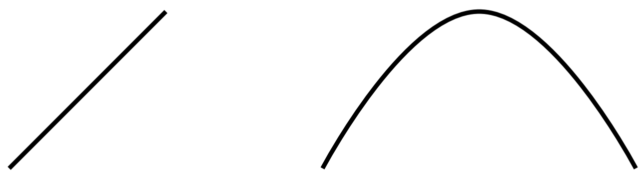

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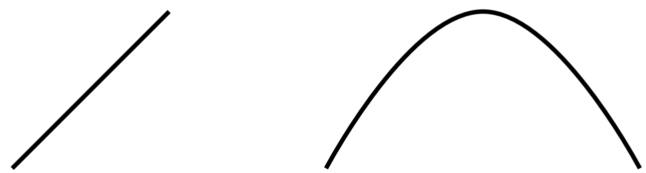
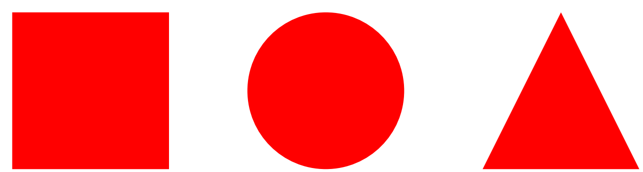


- 2-cells



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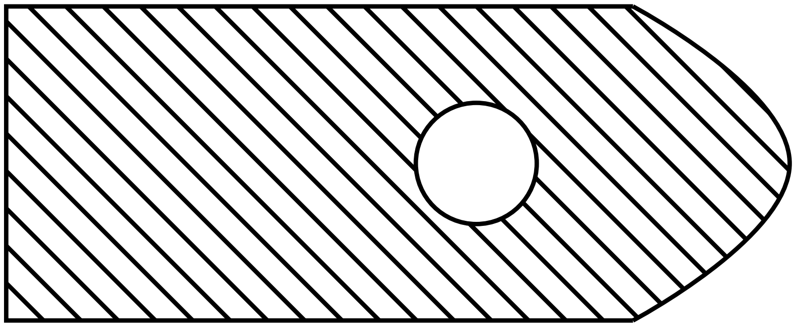

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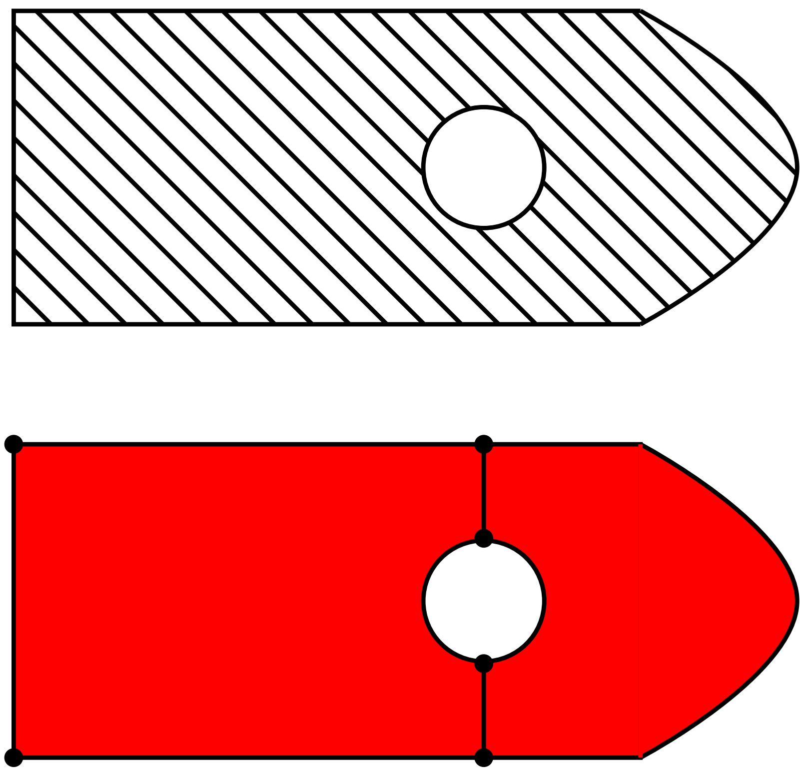

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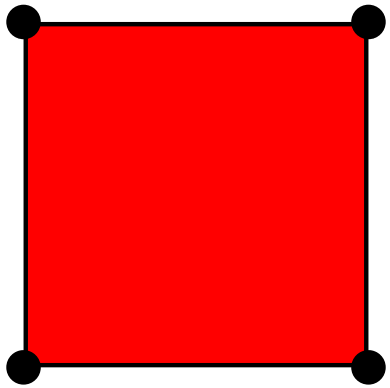

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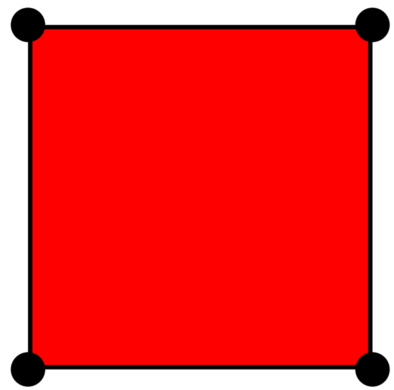

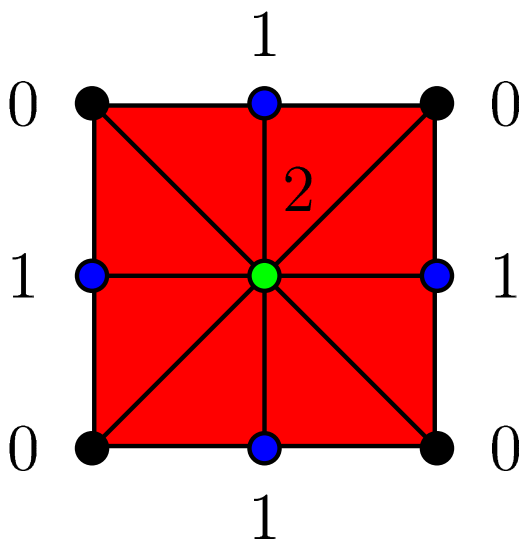

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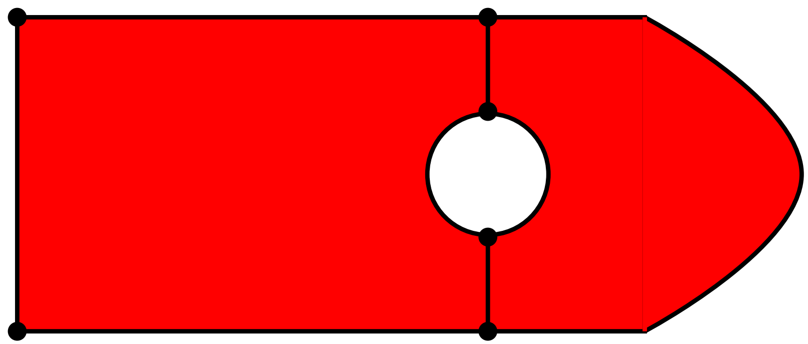

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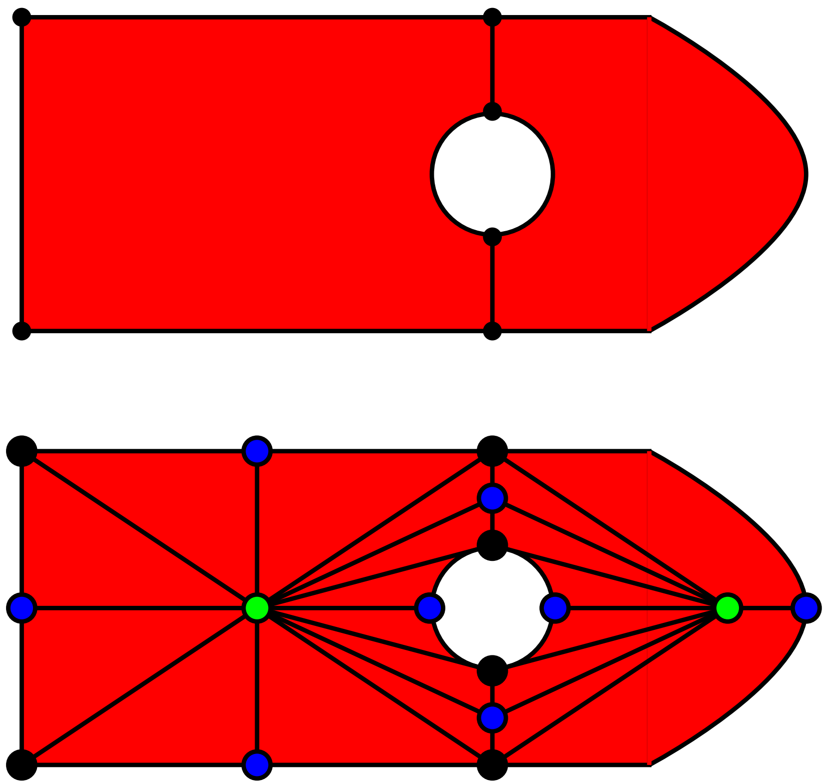

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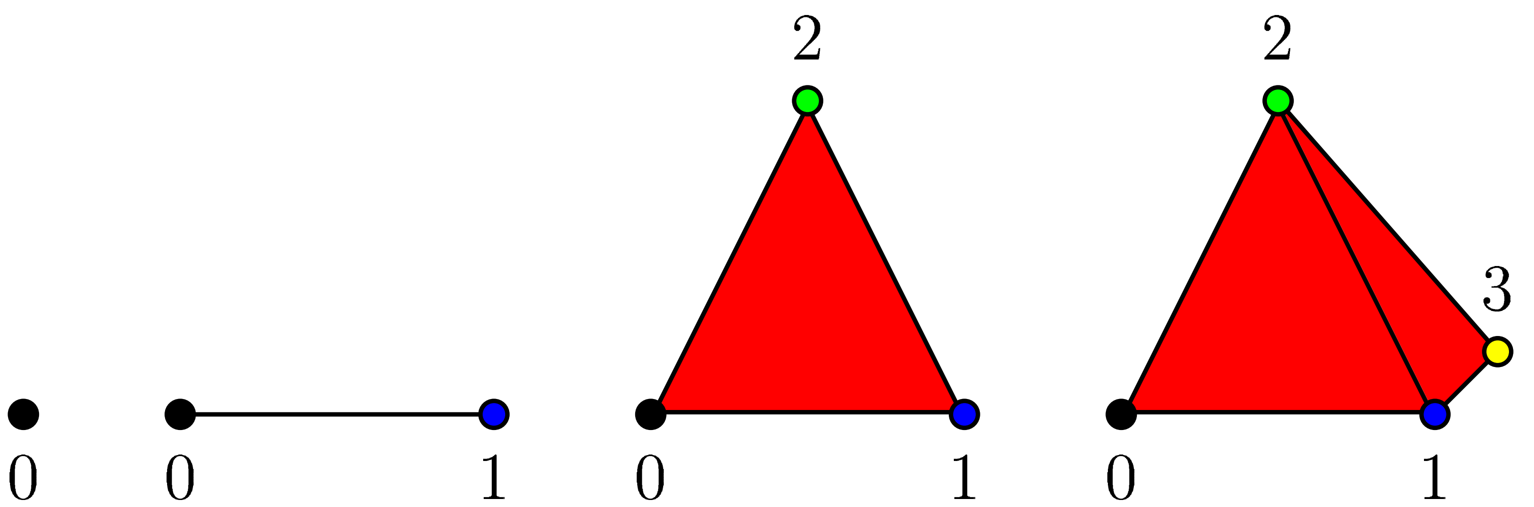

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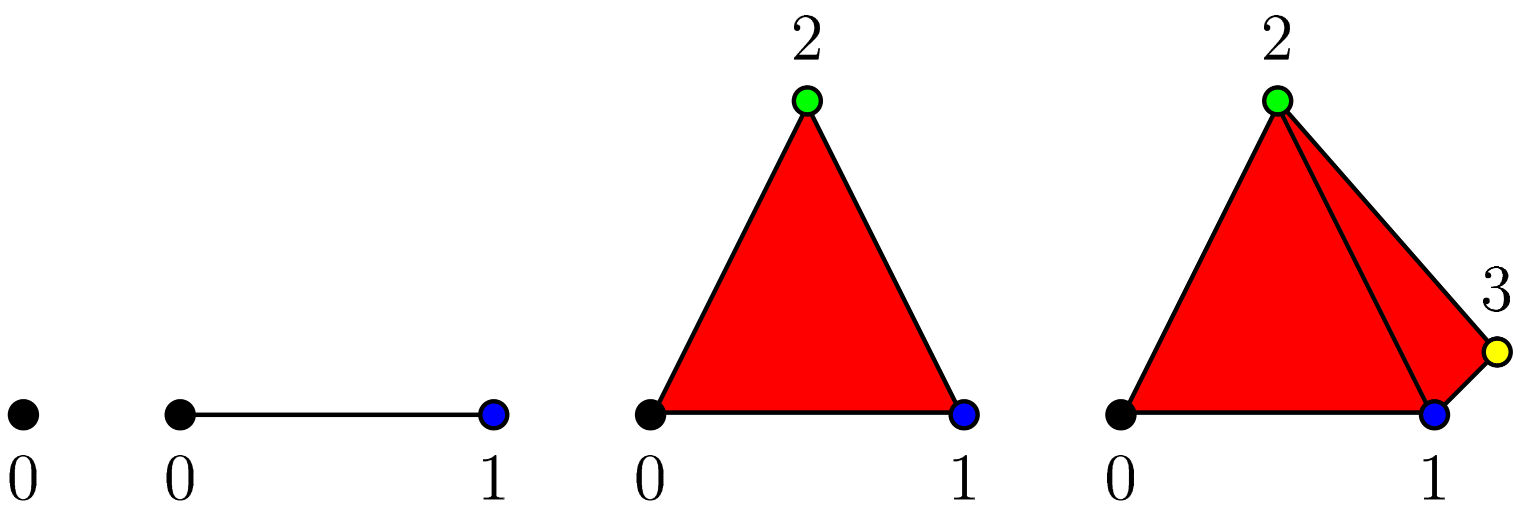

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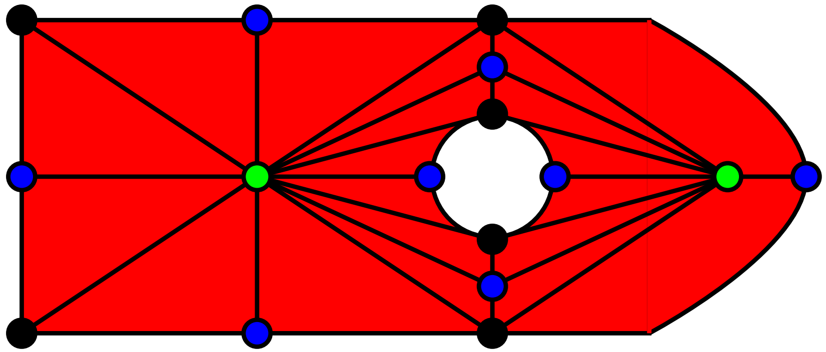
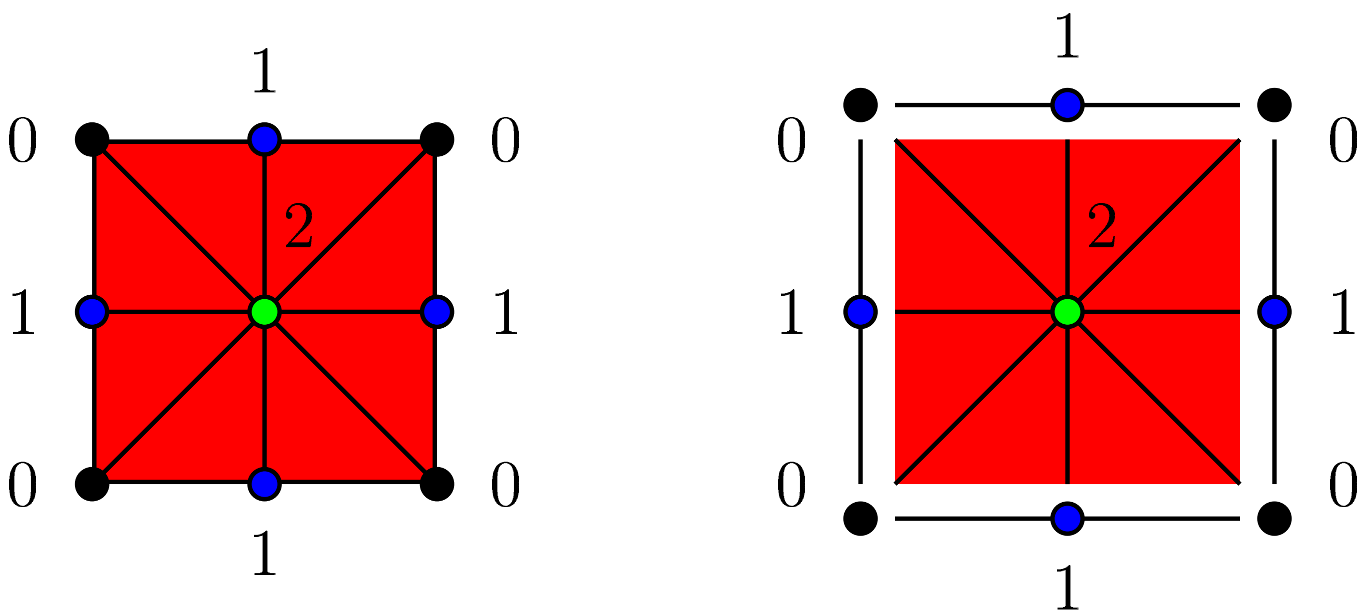
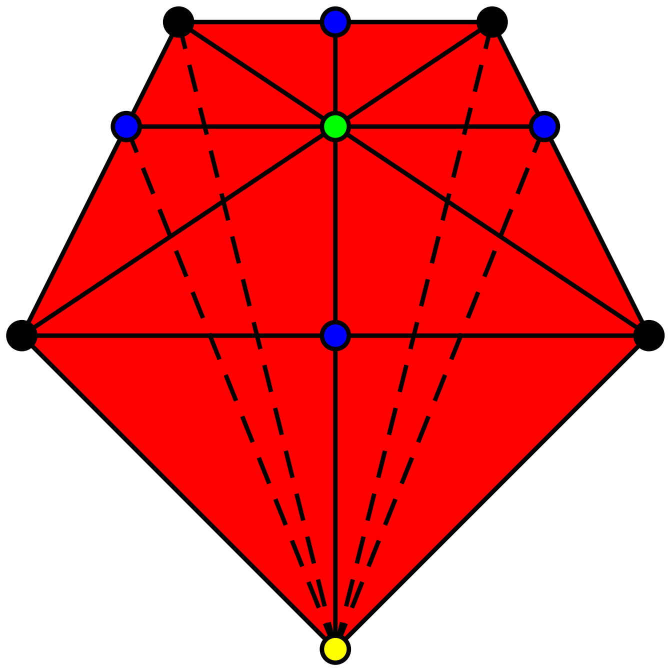
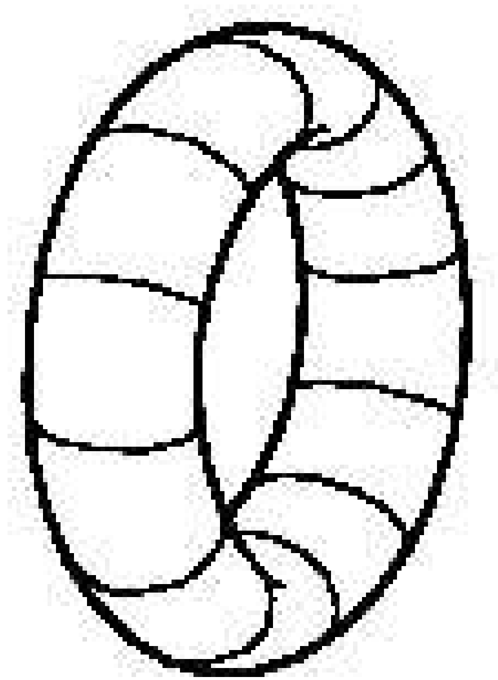


Faces of a Simplicial Set



Not a Cell Complex





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- It is recursively unsolvable for dimensions six and higher.

So what do we do?

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- Retain as many nice properties from cell complexes as possible.

Explicit Models

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- Typically represented as a DAG.

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- An abstract set S of **crosses** and three permutations P, θ and ϕ of S satisfying:

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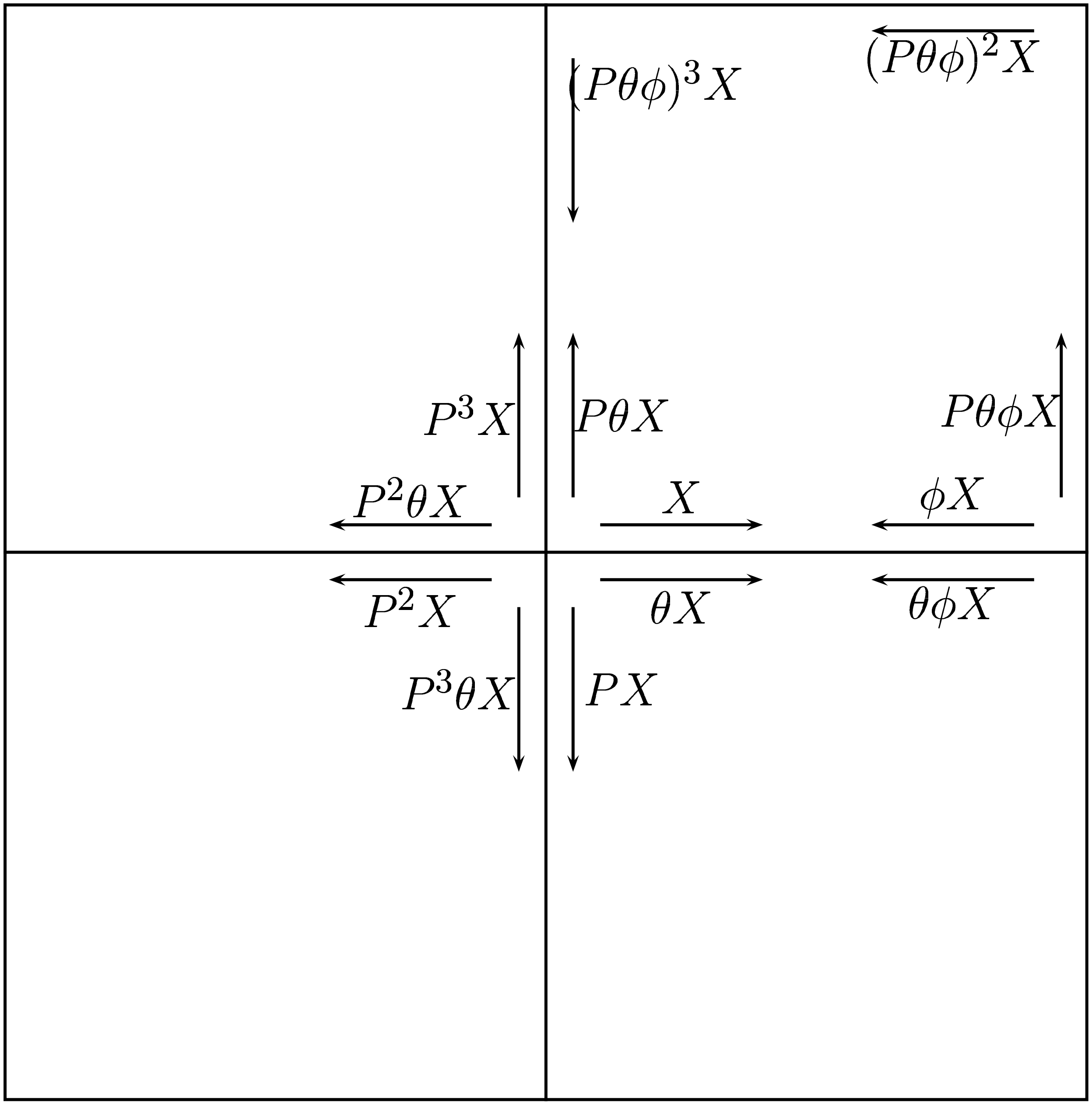
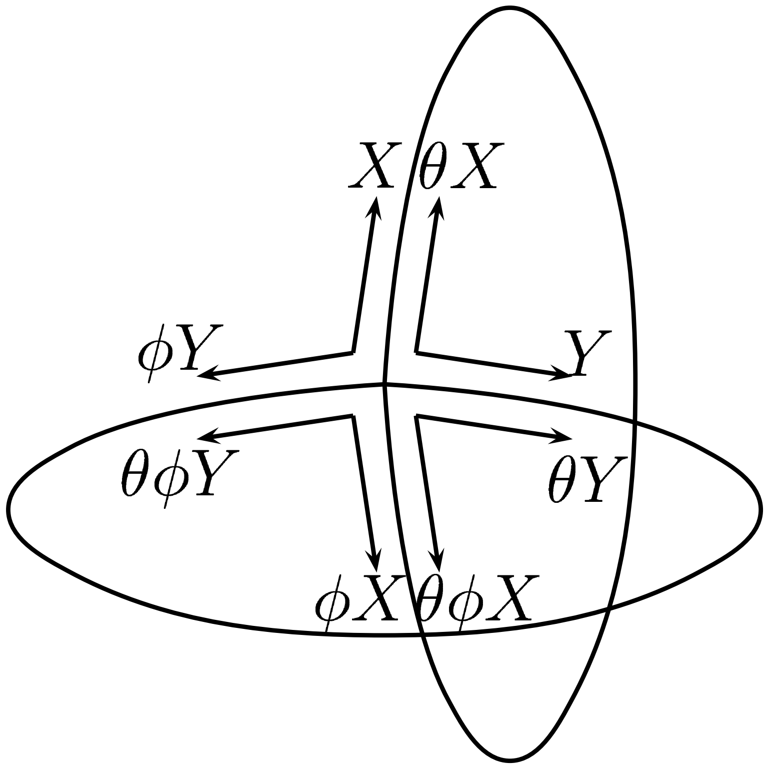
- The orbits of X and θX under P are distinct for all crosses X.

Example



$(P\theta\phi)^3X$
$(P\theta\phi)^2X$
P^3X
$P\theta X$
$P\theta\phi X$
$P^2\theta X$
X
ϕX
P^2X
θX
$\theta\phi X$
$P^3\theta X$
PX

Torus



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- Faces need not be cells.

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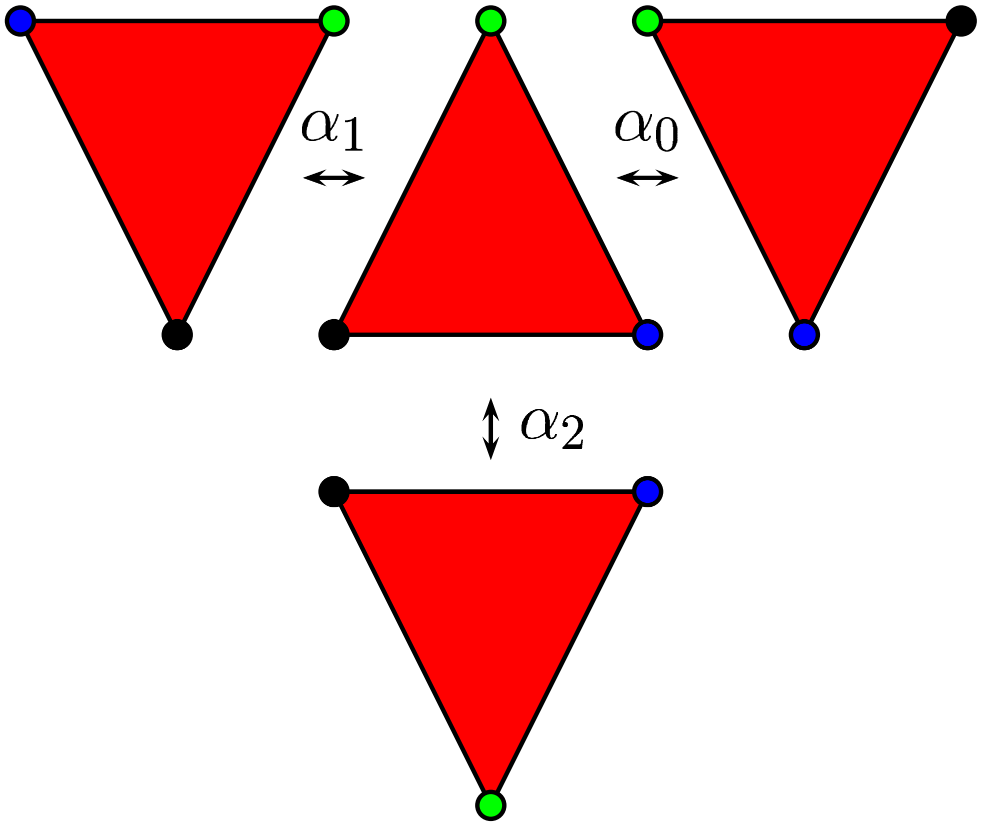
- $\alpha_i \alpha_j = \alpha_j \alpha_i$ whenever $0 \leq i < i + 2 \leq j \leq d$.

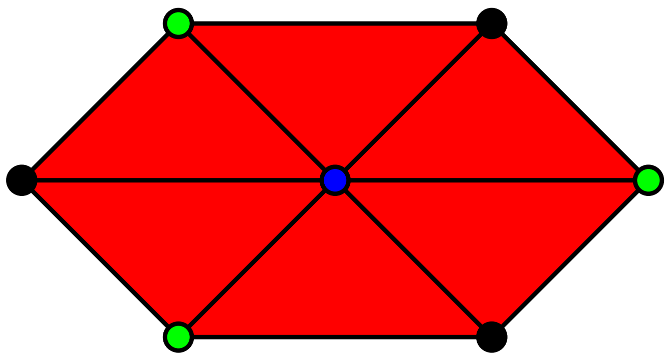
Maps and Numbered Simplicial Sets

- Every map corresponds to a numbered simplicial set.



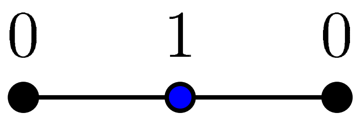
Maps and Numbered Simplicial Sets

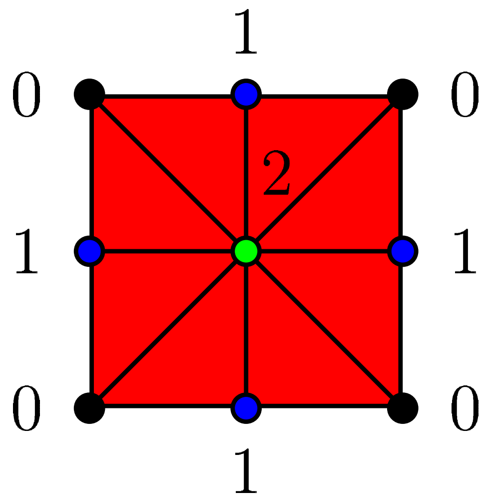
- Not every numbered simplicial set corresponds to a map.



Removing Undesirable Cases

- good configurations





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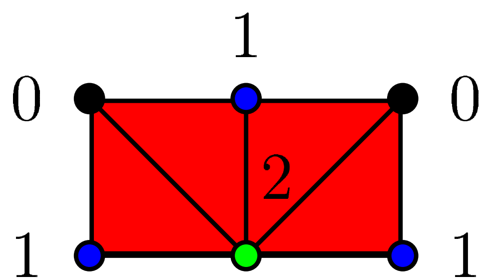

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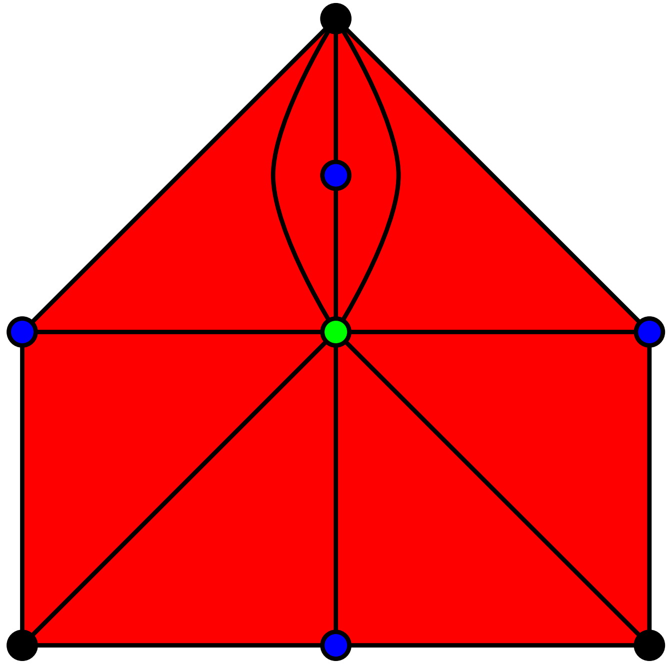

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- For any dart σ and $0 \leq i < d$, if $\alpha \in< \alpha_0, \ldots, \alpha_{i-1} >$ and $\beta \in< \alpha_{i+1}, \ldots, \alpha_d >$ then $\alpha\beta(\sigma) = \sigma$ iff $\alpha(\sigma) = \sigma$ and $\beta(\sigma) = \sigma$.