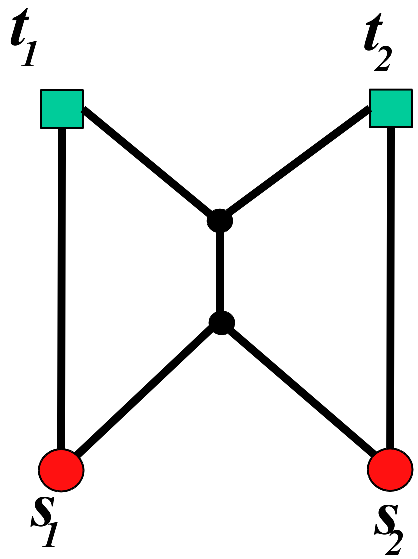
A Constant Factor Approximation Algorithm for the Multicommodity Rent-or-Buy Problem

Amit Kumar *Bell Labs*

joint work with

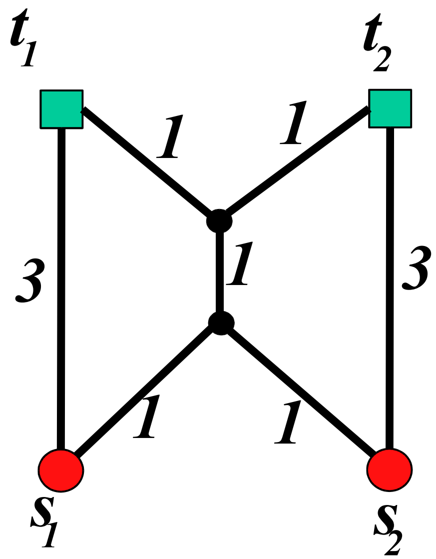
Anupam Gupta *CMU*
Tim Roughgarden *Cornell*

The Rent-or-Buy Problem



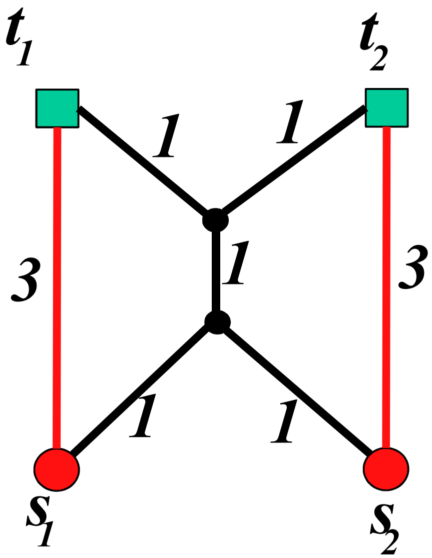
- Given source-sink pairs in a network.
- Each pair has an associated demand.
- Want to install bandwidth in the network so that the source-sink pairs can communicate.

The Cost Model

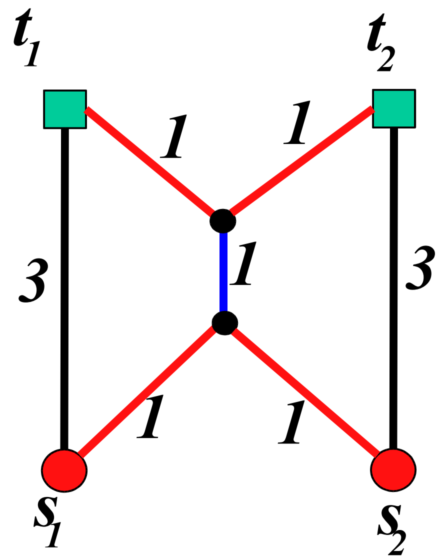


- Can rent an edge at unit cost per unit length
  - costs x units per unit length to send x unit of flow.
- Can buy an edge for M units per unit length
  - costs M units per unit length to send any amount of flow.

Example



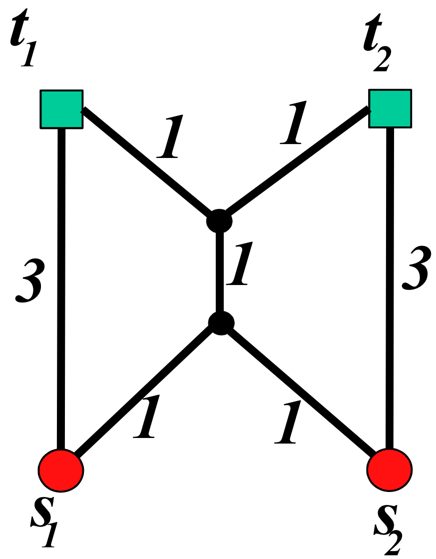
Cost = 6



Cost = 5.5

M = 1.5

Problem Definition

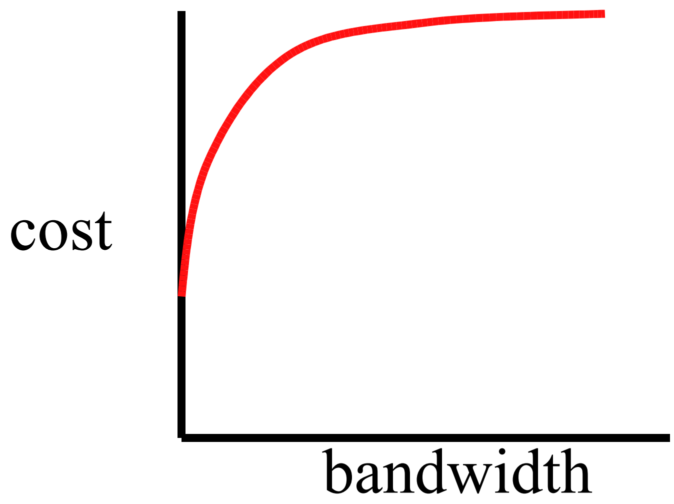


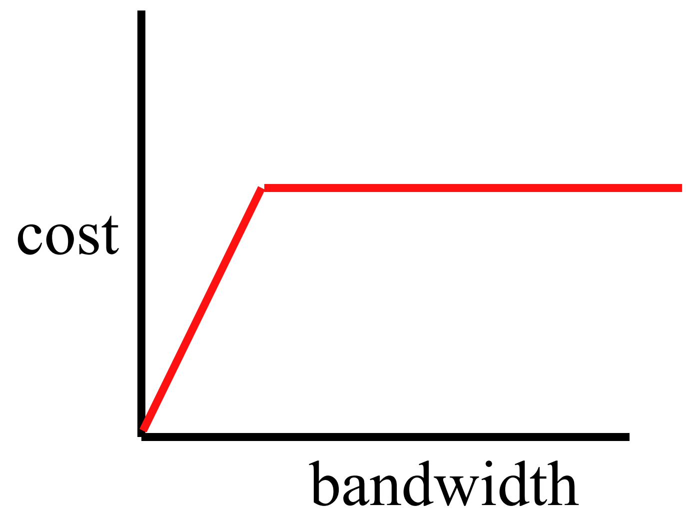
Given a graph $G = (V,E)$ and a set of source-sink pairs $\{(s_i, t_i)\}$.

Reserve enough bandwidth in the network so that s_i can send one unit of flow to t_i.

Objective : Minimize the total cost incurred.

Economies of Scale



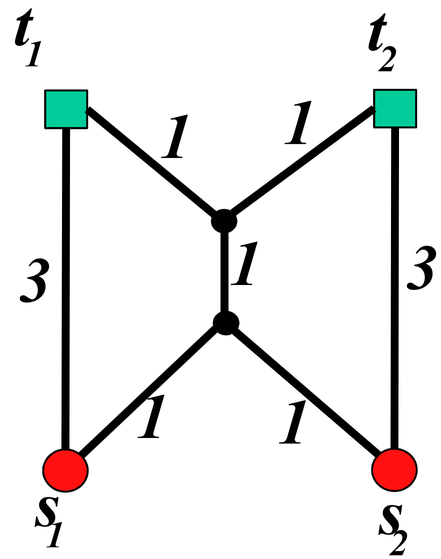


Edge cost a concave function of the amount of flow sent.

Rent-or-buy :

- a non-trivial special case of this problem.
- has applications to multi-commodity connected facility location problem, multi-commodity maybe-cast problem [KM '00].

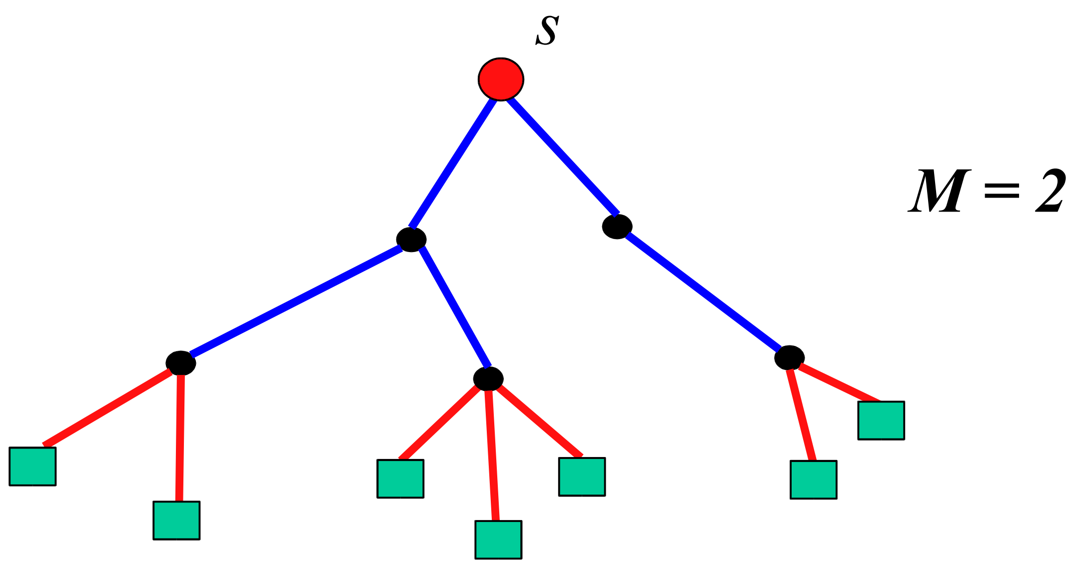
Special Cases



- ***M = 1*** : Always buy edges, equivalent to generalized Steiner tree problem (NP-hard).
- ***M = infinity*** : rent shortest source-sink paths.

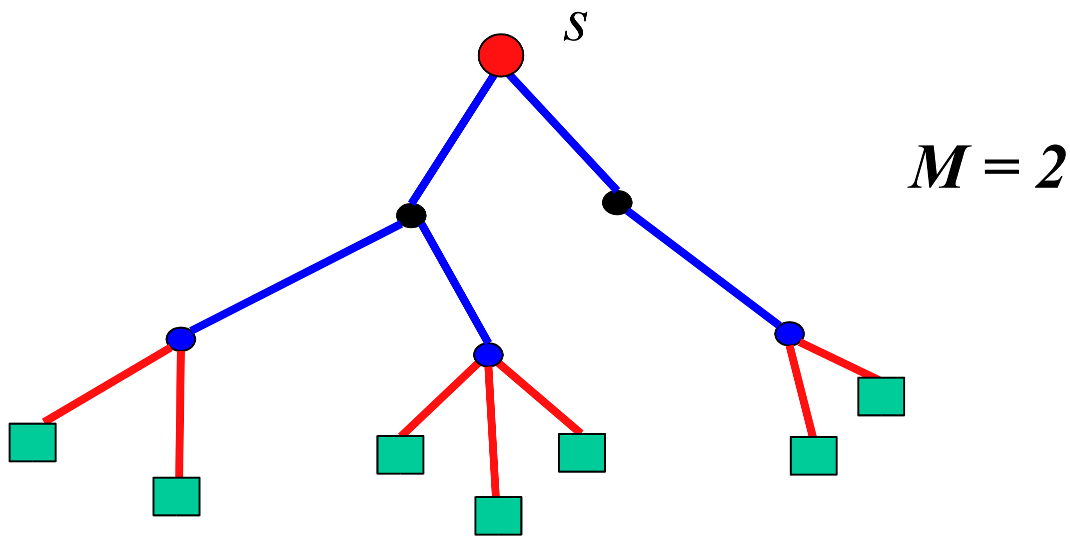
Our result : first constant factor approximation algorithm for the rent-or-buy problem.

Single Source Case



- Edges on which bandwidth reserved form a tree.
- Monotonicity Property : the rented edges on any source sink path form a connected subpath.

Single Source case



First constant factor approx. algorithm :

Karger and Minkoff FOCS 2000

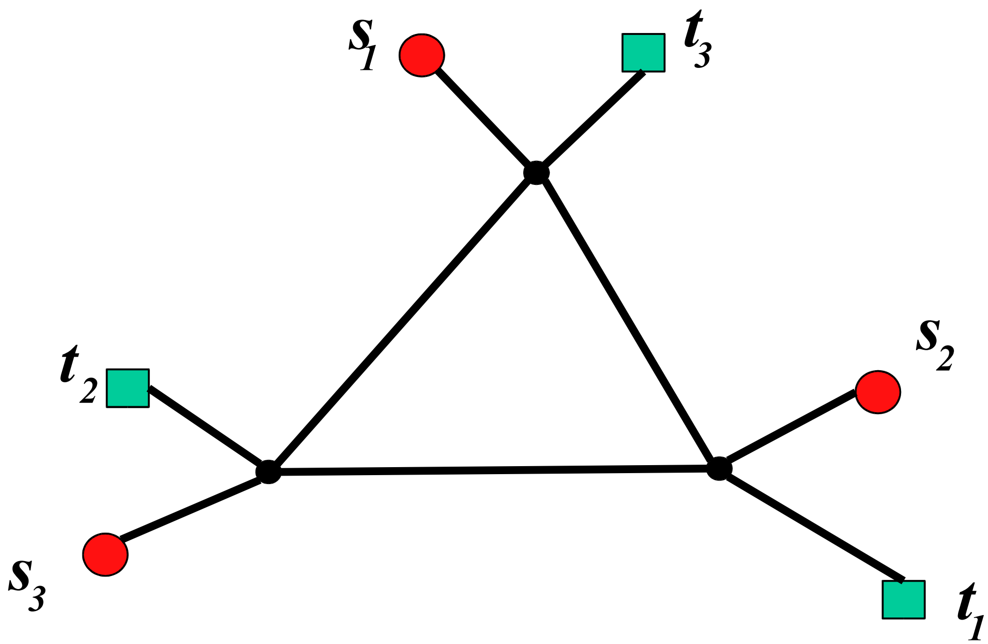
Constant factor improved to

9.001 : Gupta,Kleinberg,Kumar,Rastogi,Yener STOC 2001

4.55 : Swamy and Kumar APPROX 2002

3.51 : Gupta, Kumar and Roughgarden STOC 2003

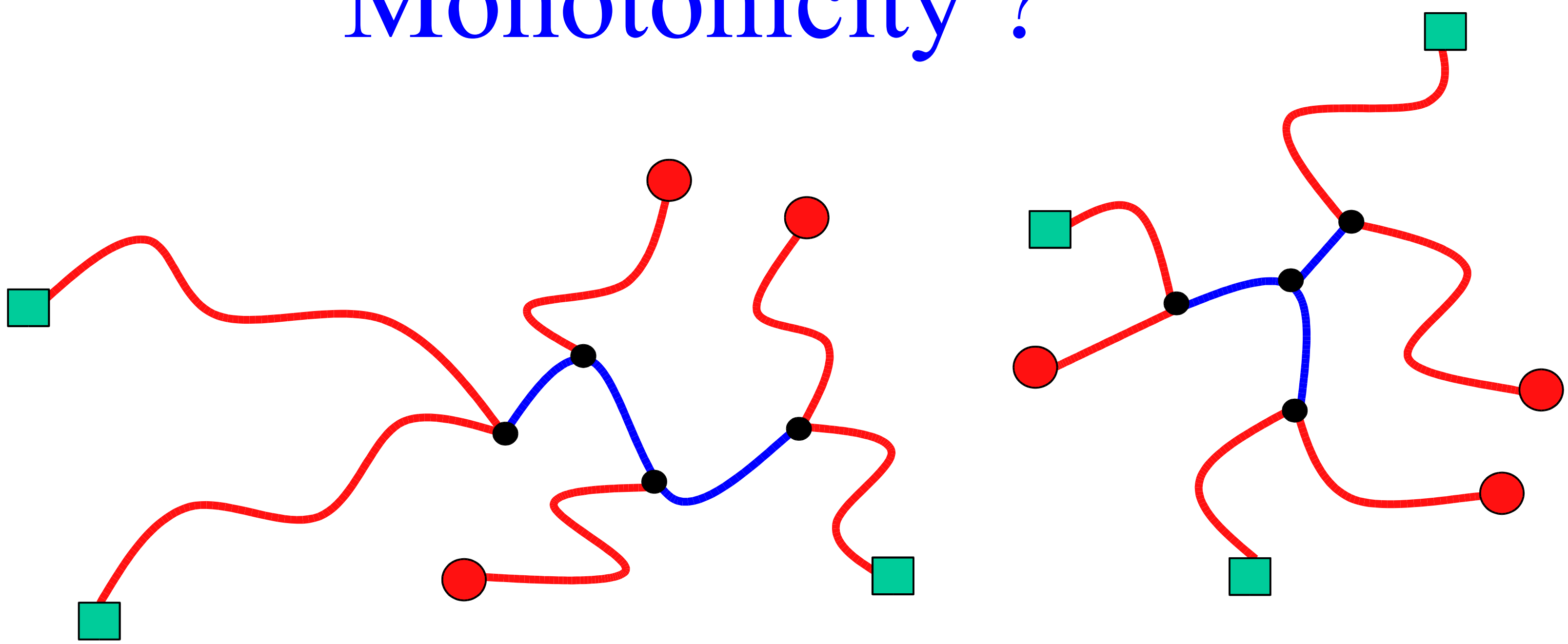
Rent-or-Buy Problem



What properties of single source case carry over to this problem ?

- The set of edges used need not be a tree in any optimal solution.
- The set of edges that an optimal solution buys induce a forest.

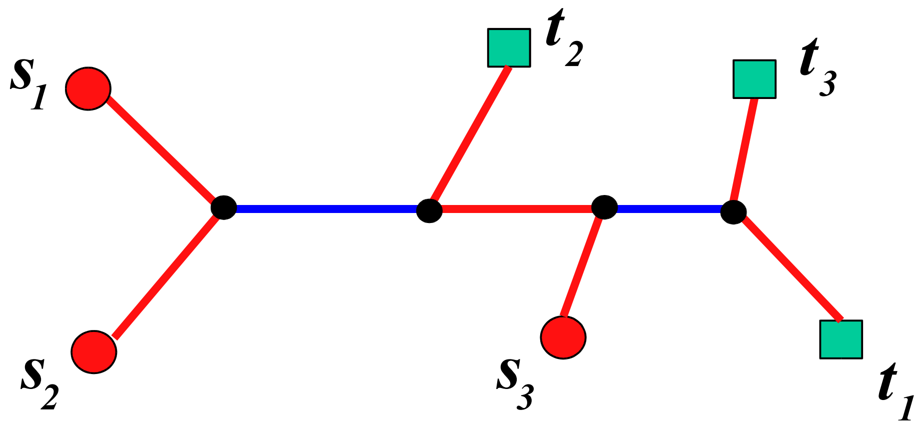
Monotonicity ?



The definition of monotonicity must be symmetric with respect to sources and sinks.

The set of bought edges in any source to sink path forms a connected subpath.

Monotonicity

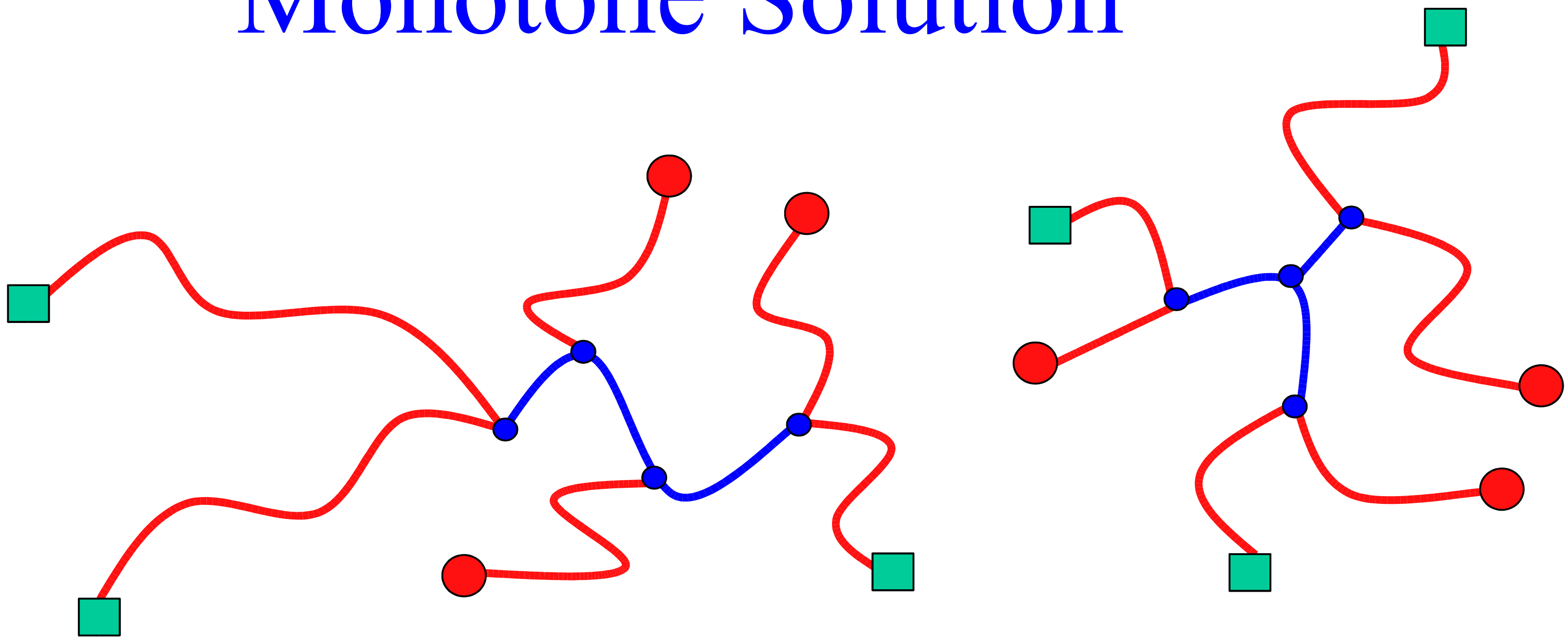


$M = 1.5$

Optimal solution may not be monotone.

Theorem : There is a monotone solution whose cost is at most twice the cost of the optimal solution.

Monotone Solution

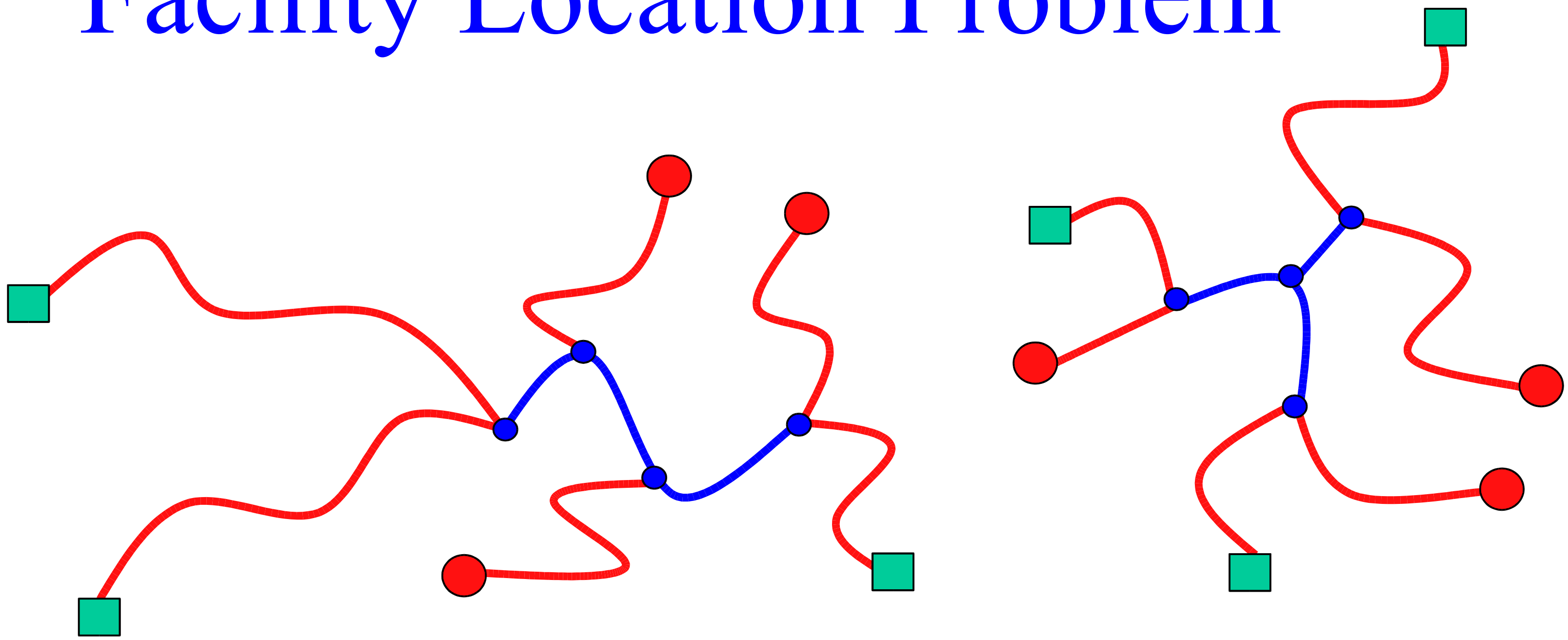


Look for optimal monotone solution only.

Enough to know the component of bought edges.

We can think of the vertices in these components as *facilities*.

Facility Location Problem



Want to find facility nodes *F*.

Each source or sink attaches to a node in *F* and pays the shortest distance to this node.

For any *(s,t)* pair, the facilities to which *s* and *t* attach must be connected by bought edges – pay *M* times the length of edges bought.

Integer Programming Formulation

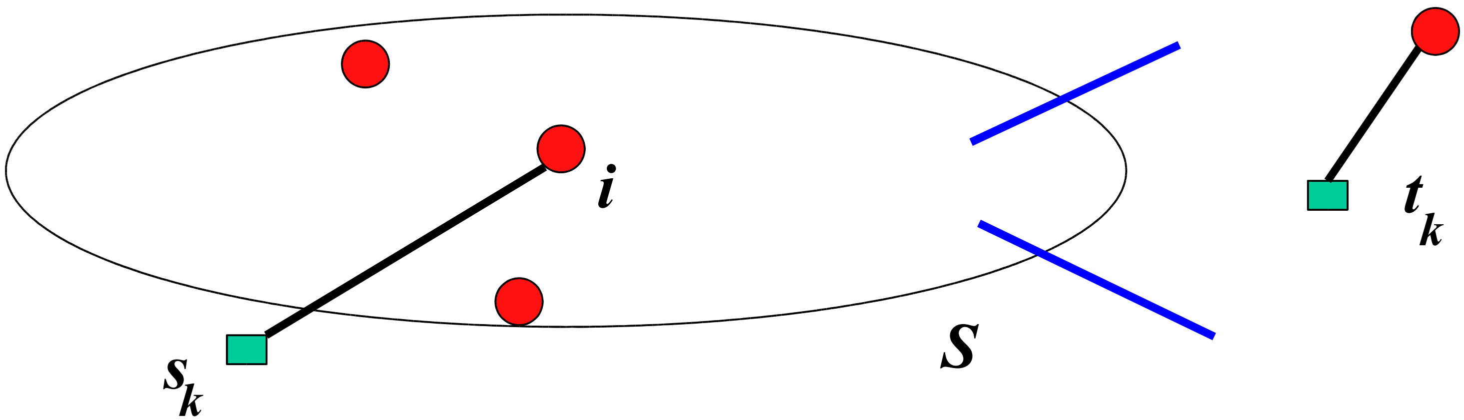
Decision variables with values either *0* or *1*.

- x_{ij} : *1* if demand *j* assigned to *i*.
- z_e : *1* if the edge *e* is bought.

Objective function :

$$\min \sum_{j \in D} \sum_{i \in V} x_{ij} d(i,j) + M \sum_{e \in E} c_e z_e$$

Constraints



$$\sum_{i \in V} x_{ij} = 1 \text{ for all } j \in D$$

$$\sum_{e \text{ out of } S} z_e \geq \sum_{i \in S} x_{is_k} - \sum_{i \in S} x_{it_k} \text{ for all } S \subseteq V$$

$$\sum_{e \text{ out of } S} z_e \text{¿} \sum_{i \in S} x_{it_k} - \sum_{i \in S} x_{is_k} \text{ for all } S \subseteq V$$

$$0 \leq x_{ij}, z_e \text{¿} 1$$

Related Work

- The LP follows along the line of the LP given for the single source special case by [GKKRY 01].
- They round the fractional solution with a constant factor loss.
- This technique does not extend to our problem.
- We use primal-dual method to show that the integrality gap of the LP is a constant.
- The primal-dual algorithm for the single source case given by [SK02] -- our algorithm derives intuition from this though requires several non-trivial ideas.

Dual for the rent-or-buy problem

$$\max \sum_j \alpha_j$$

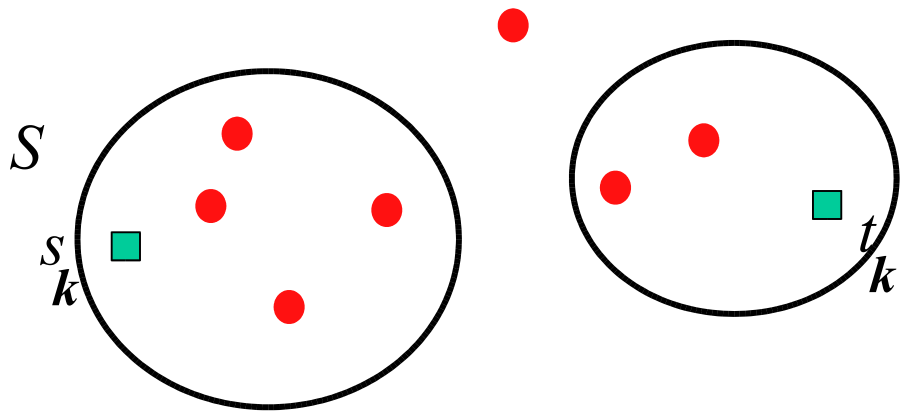
$$\alpha_{s_k} - \sum_{S:i \in S} y_{S,s_k} + \sum_{S:i \in S} y_{S,t_k} \le c_{i,s_k} \quad \text{for all } i \text{ and demands } (s_k, t_k)$$

$$\alpha_{t_k} - \sum_{S:i \in S} y_{S,t_k} + \sum_{S:i \in S} y_{S,s_k} \le c_{i,t_k} \quad \text{for all } i \text{ "} and\ demands\ (s_k, t_k)$$

$$\sum_j \sum_{S:e \in \delta(S)} y_{S,j} \le Mc_e \quad \text{for all edges } e$$

$$\alpha, y \ge 0$$

Dual variables



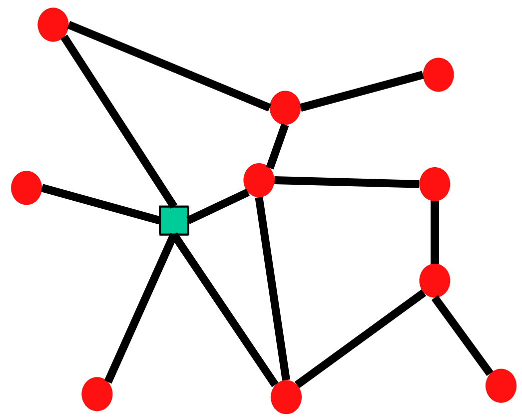
For each demand j we associate a set $S(j)$.

If $\alpha_j > c_{ij}$, then $S(j)$ contains i, $S(j)$ may contain other nodes .

When we raise α_j, we also raise $y_{S(j)\,j}$ at the same rate.
the set $S(j)$ grows as α_j grows.

The sets $S(s_k)$ and $S(t_k)$ are always disjoint.

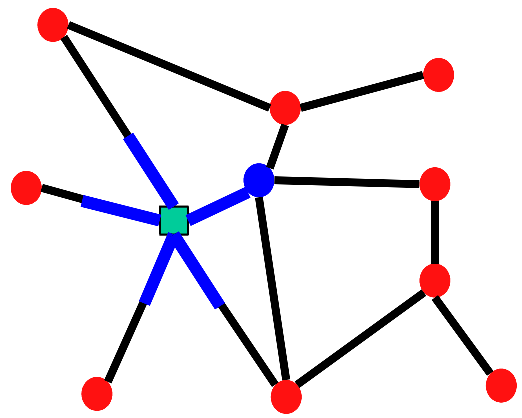
Growing dual variables



We raise α_j in a continuous way.

Can think of *S(j)* as a sub-graph which grows in a continuous way.
Can always choose to add new vertices to *S(j)*.

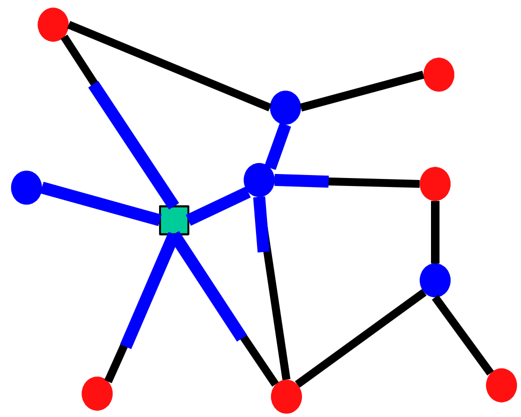
Growing dual variables



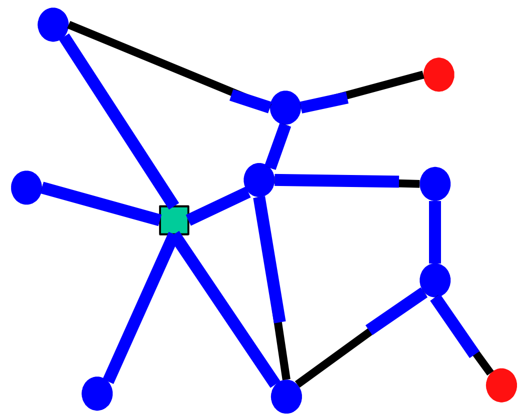
We raise α_j in a continuous way.

Can think of *S(j)* as a sub-graph which grows in a continuous way.
Can always choose to add new vertices to *S(j)*.

Growing dual variables

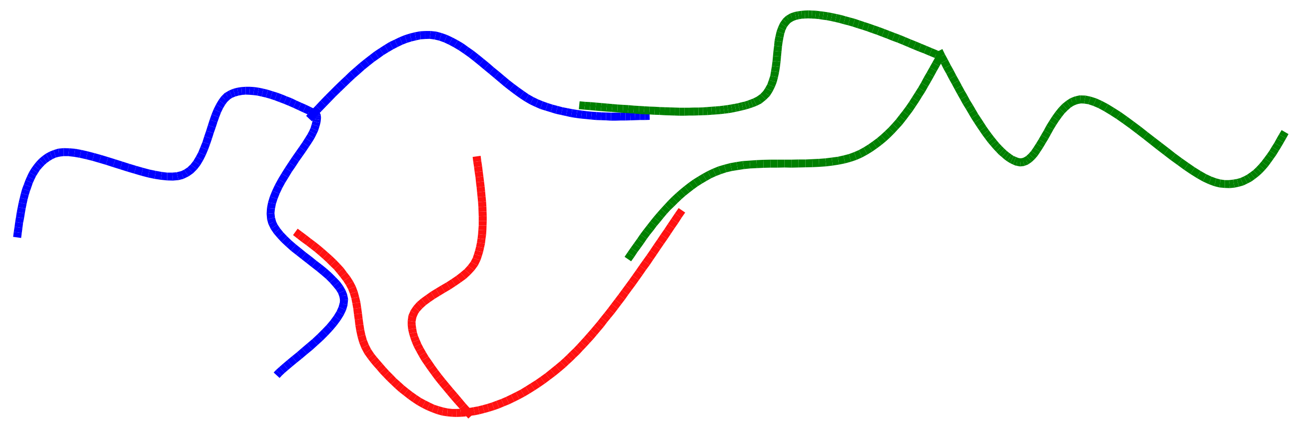


Growing dual variables



Dual Feasibility

$M = 2$



For any edge e, there will be at most M demands j such that e will ever appear in the set $\delta(S(j))$.

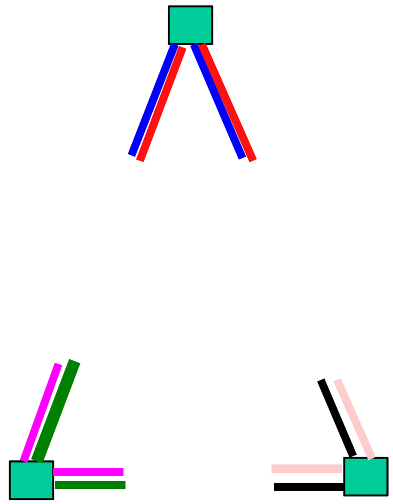
Primal-dual Algorithm

Special case :

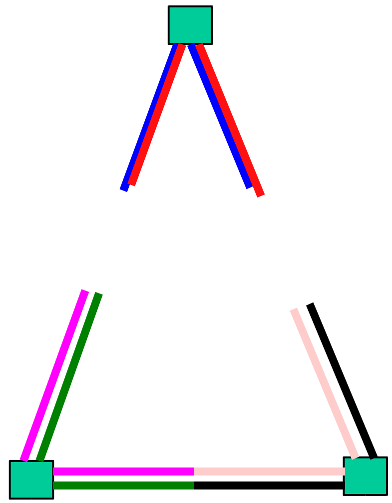
- All demands occur in groups of *M*.
- All demand pairs have the same source.

Should be able to simulate Steiner tree primal-dual algorithm.

Primal-dual Algorithm

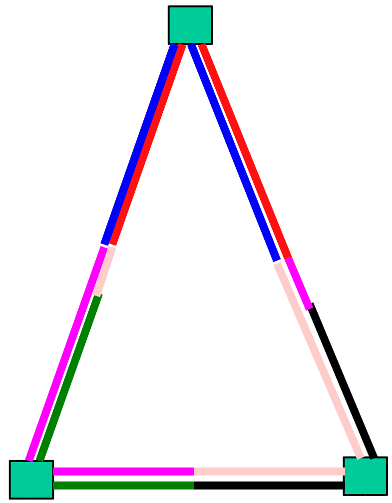


Primal-dual Algorithm



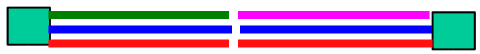
When two components merge, raise only *M* dual variables in it.

Primal-dual Algorithm



Problems

- Suppose demands do not have a common source.



If (s,t) is a demand pair, can raise their dual variables as long as their sub-graphs are disjoint.

Even if demands occur in groups of M, this may not be the case when we raise the dual variables.

- Demands may not occur in groups of M.
  – raise dual variables so that they form groups of size M.

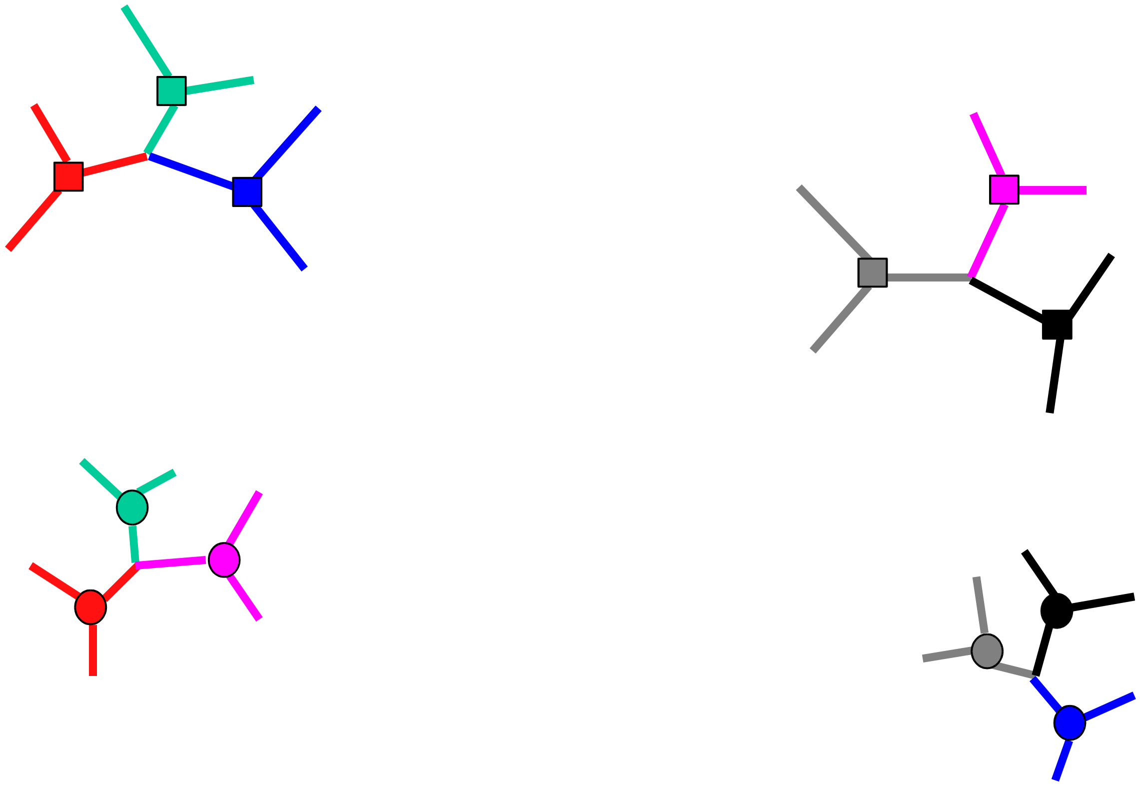
Primal-dual Algorithm

- Raise the dual variables such that clusters of at least M demands form.
- Collapse the clusters into single nodes and simulate the Steiner forest primal-dual algorithm as long as each component has at least M nodes.

The algorithm alternates between the two steps.

Primal-dual Algorithm

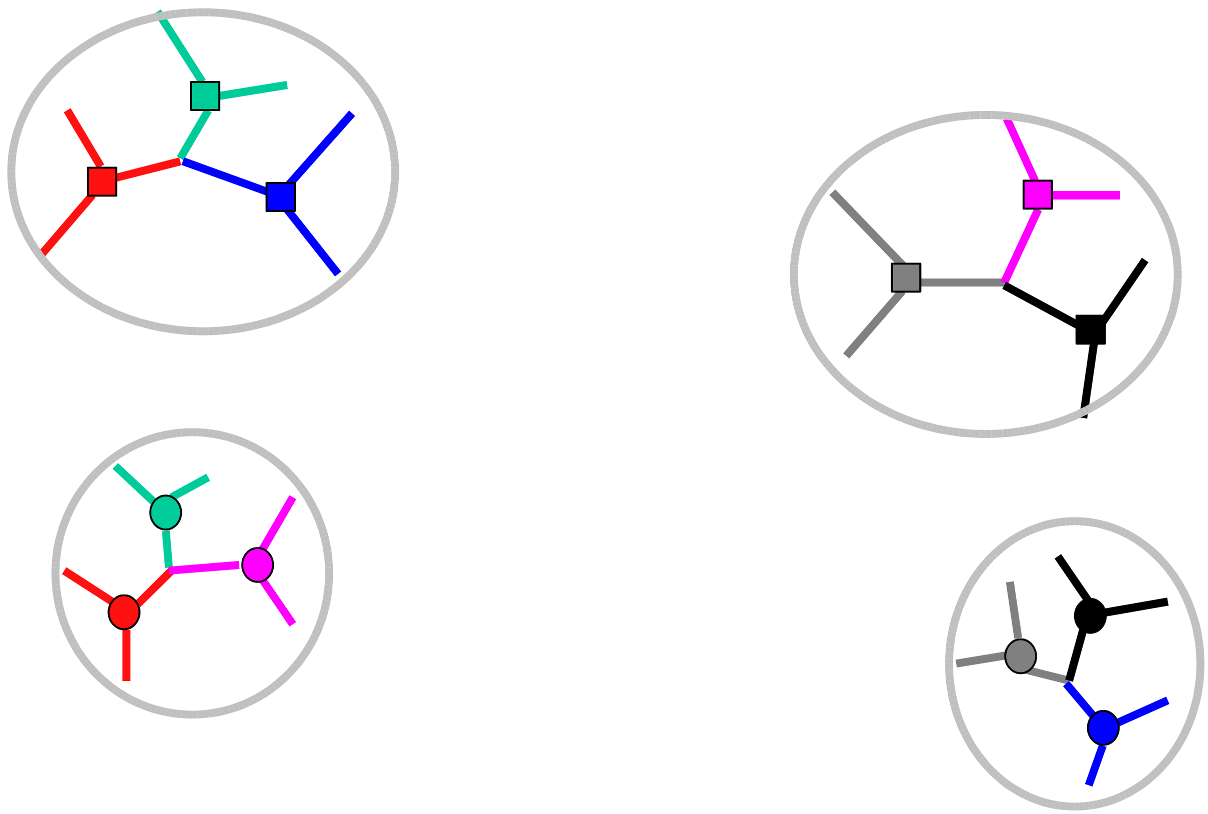
$M = 3$



Consolidate demands into groups of M.

Primal-dual Algorithm

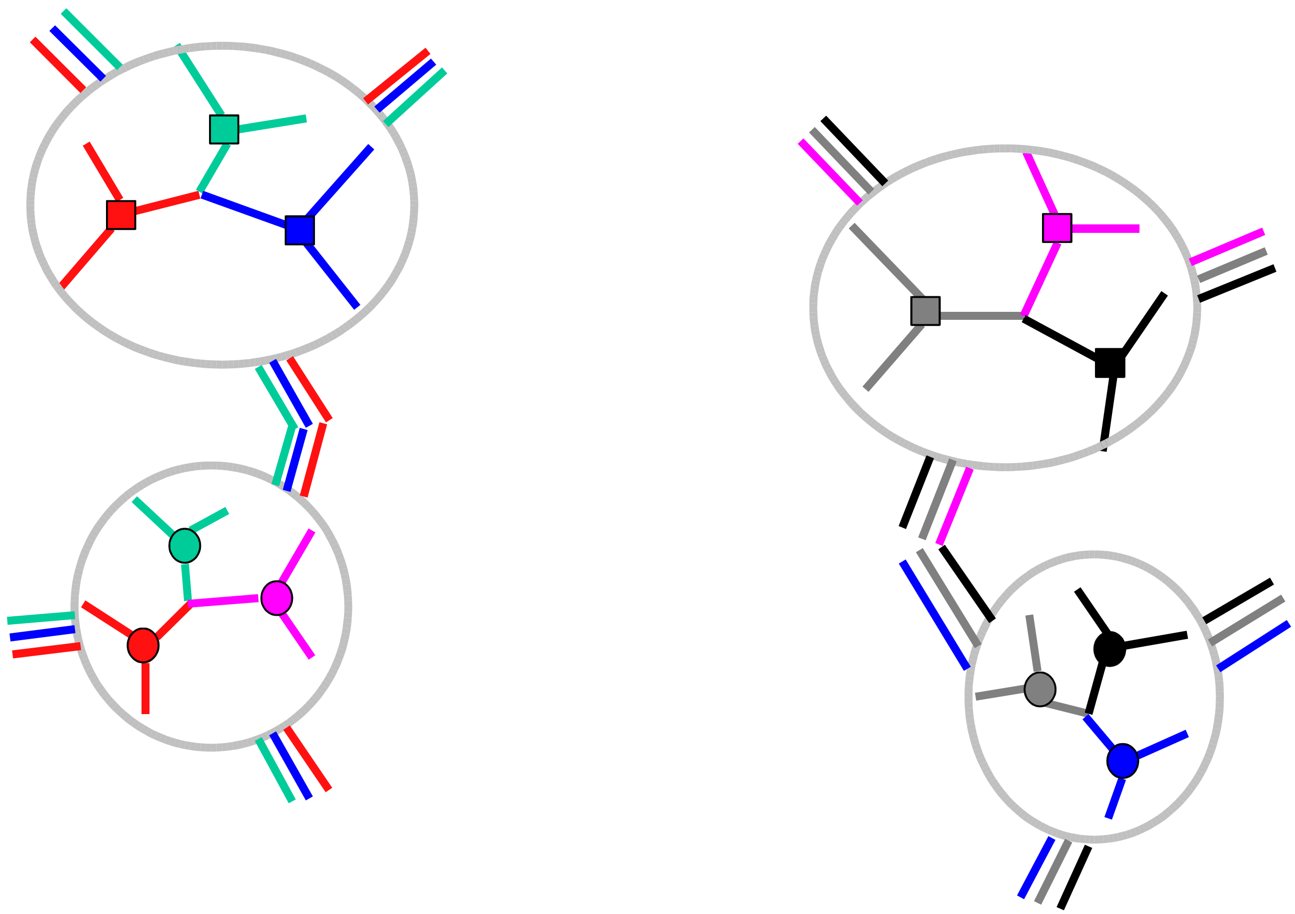
$M = 3$



Collapse clusters into single nodes.

Primal-dual Algorithm

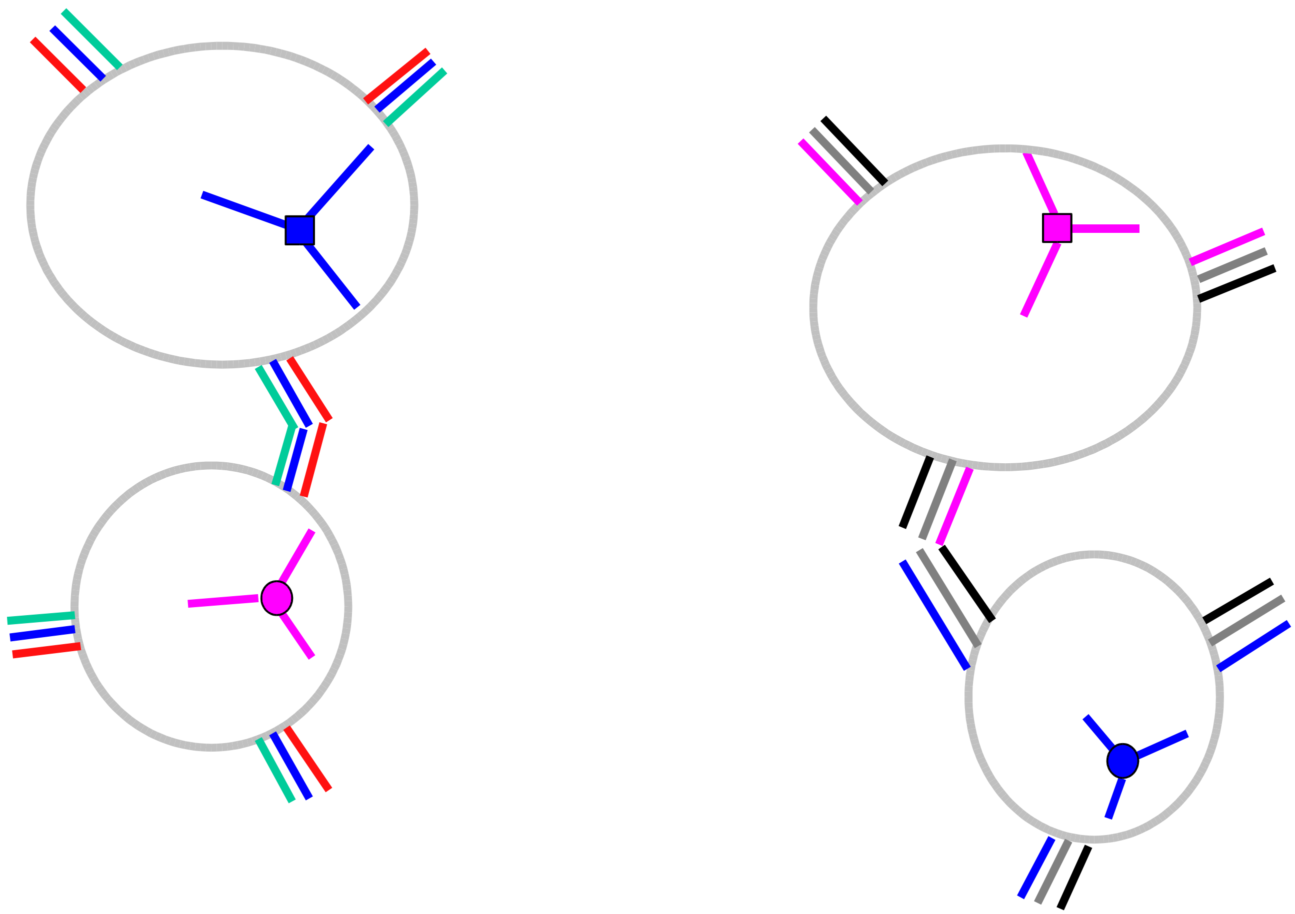
$M = 3$



Simulate primal-dual algorithm for the Steiner forest problem.

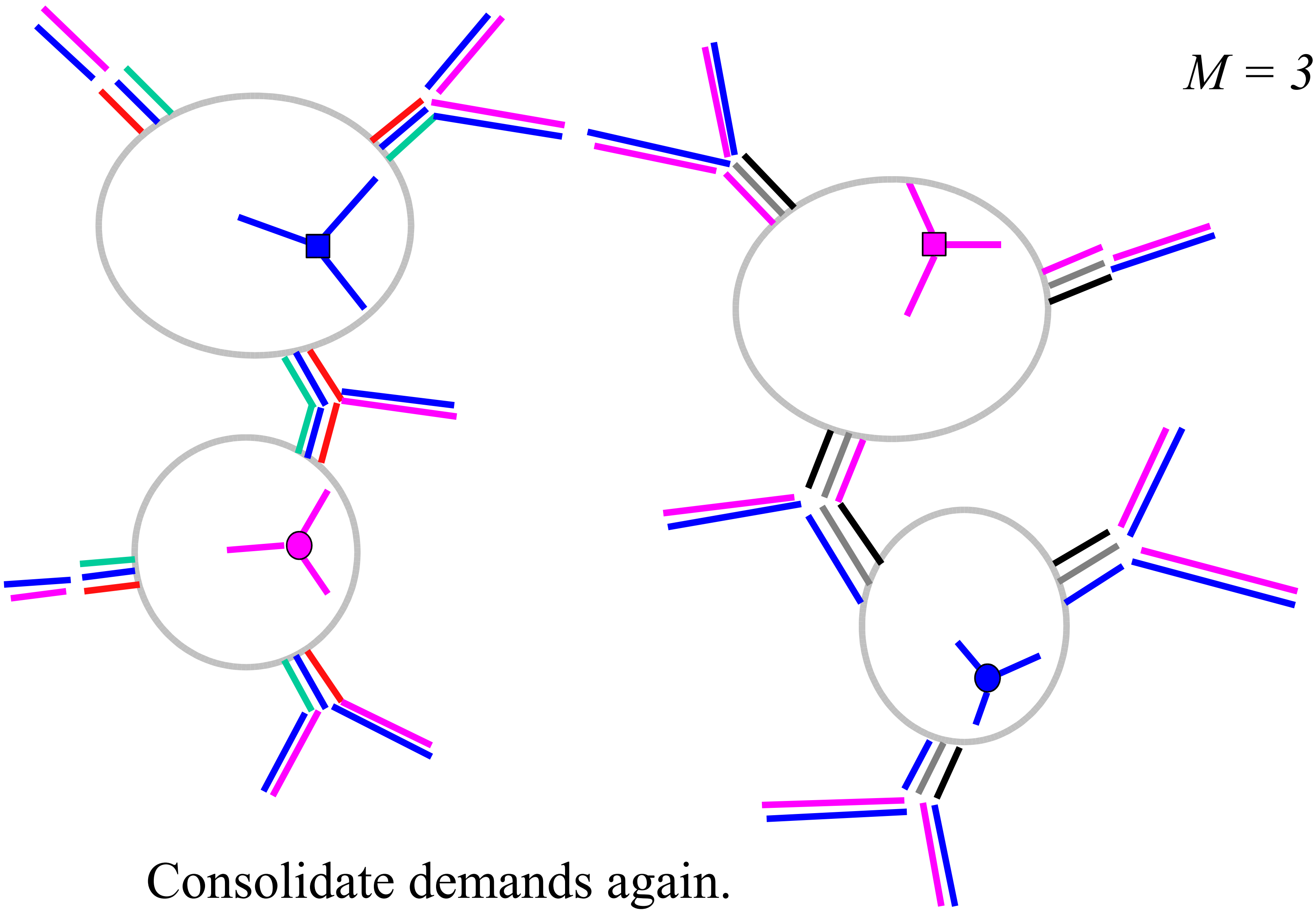
Primal-dual Algorithm

$M = 3$



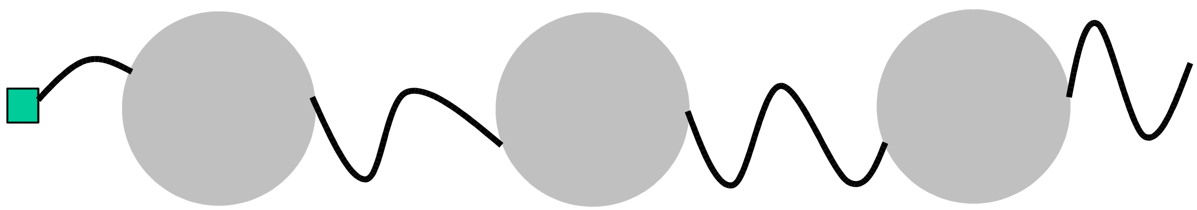
Cancel demands.

Primal-dual Algorithm



Consolidate demands again.

Primal-dual algorithm

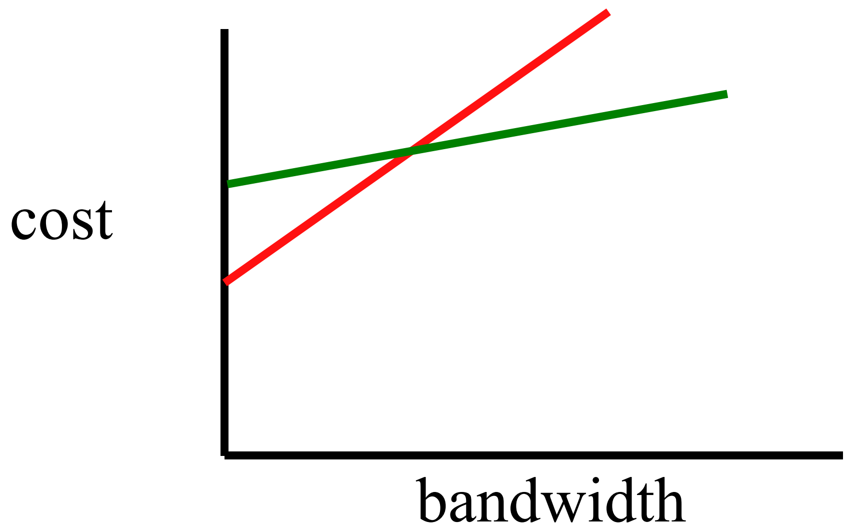


In a given phase, many edges may become *unusable* because of previous phases.

Need to bound the effect of previous phases on the current phase

-- make sure that the dual variables increase by an exponential factor in successive phases.

Open Problems



- buy-at-bulk network design :
  have many different types of cables, each has a fixed cost and an incremental cost.

Single source buy-at-bulk : constant factor approximation algorithm known [Guha Meyerson Munagala '01].

Open Problems

- Buy-at-bulk : even for single cable nothing better than logarithmic factor known [Awerbuch Azar '97].

- Single source rent-or-buy : what if different edges have different buying costs ?

  Nothing better than logarithmic factor known [Meyerson Munagala Plotkin '00].