Covering Graphs using Trees and Stars

G. Even N. Garg J. Könemann R. Ravi A. Sinha

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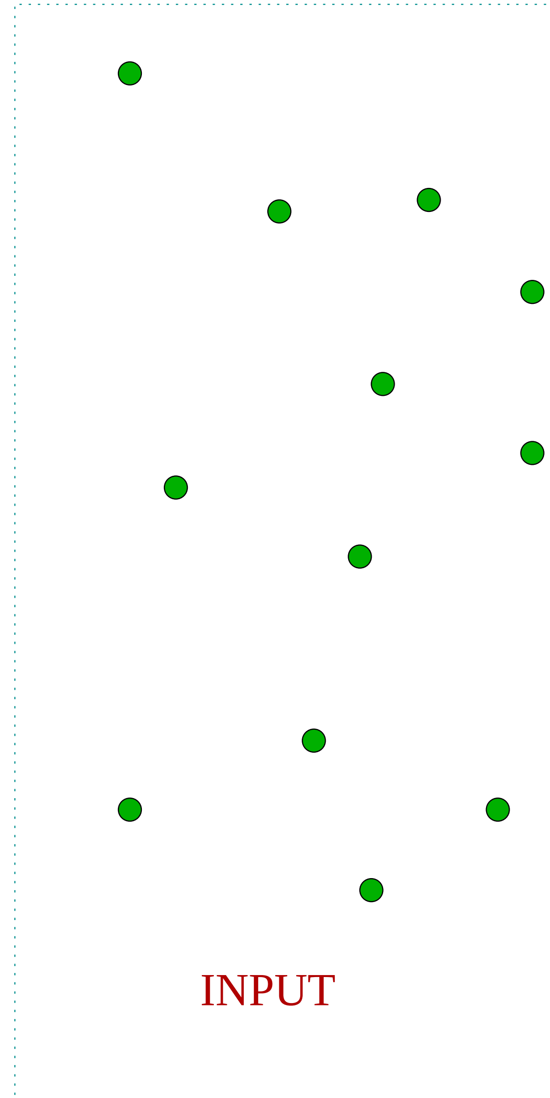
- At 8 am, each nurse begins her “morning round” of patients under her care.

- Morning round ends when *all* nurses have returned to their bases.

- Objective: Assign patients to nurses so that morning rounds end ASAP.

Problem definition

- Input: Graph $G = (V, E)$, edge weights w, integer k.



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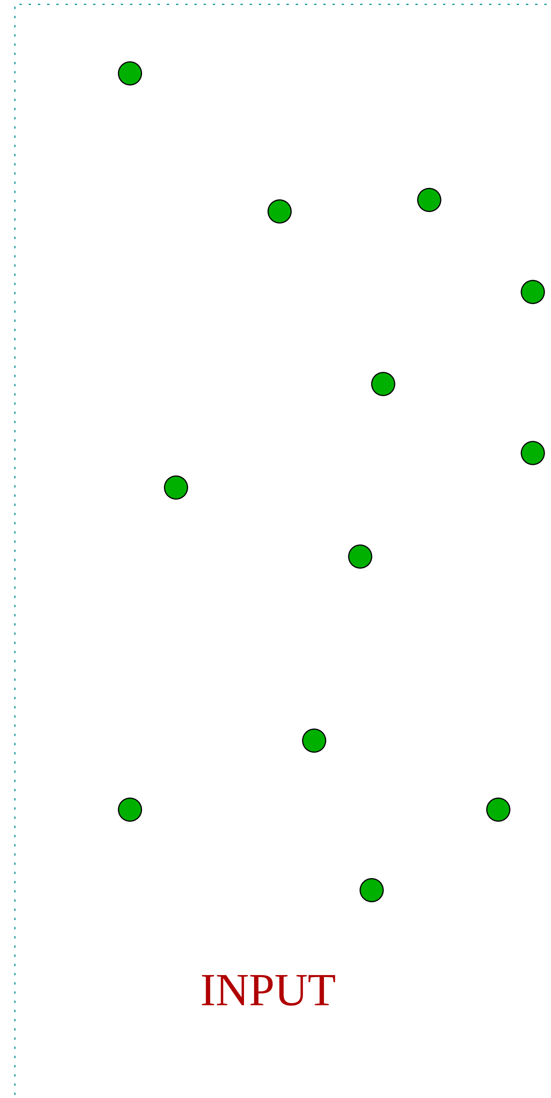

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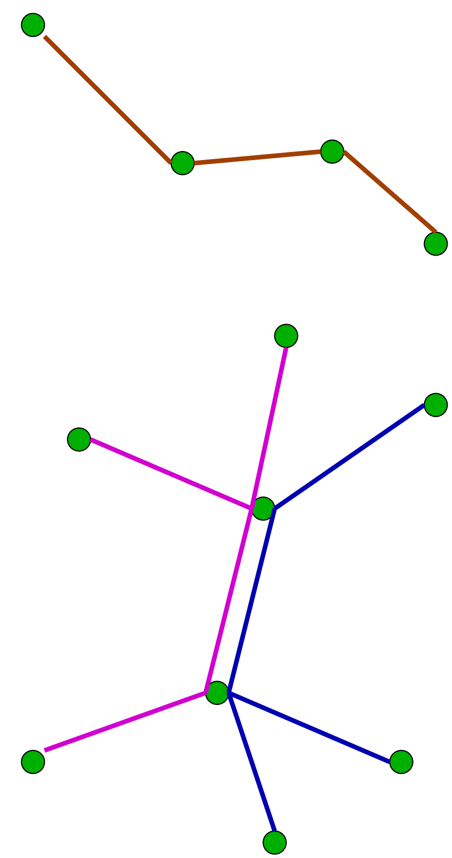

FEASIBLE SOLUTION

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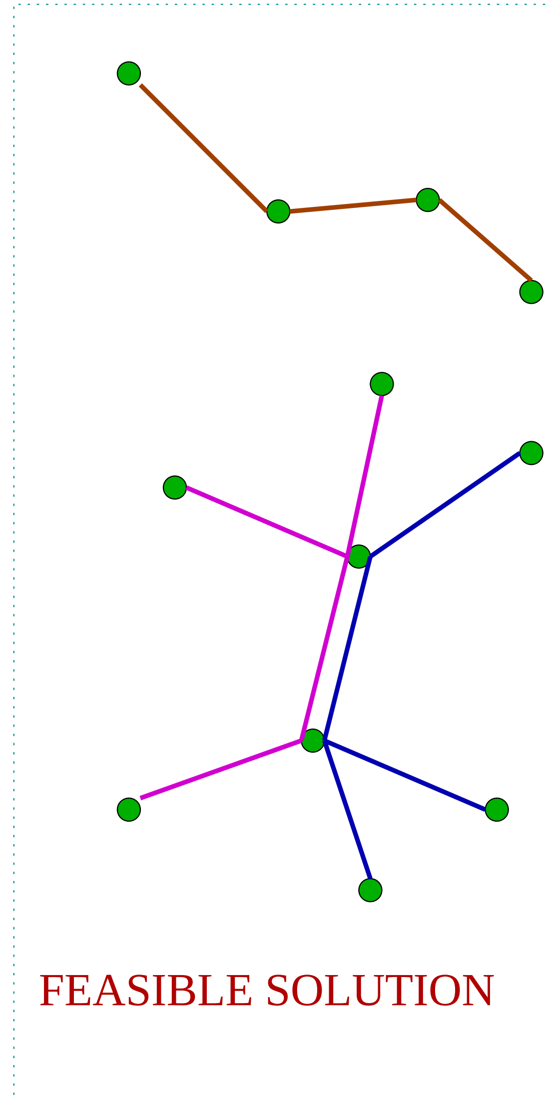

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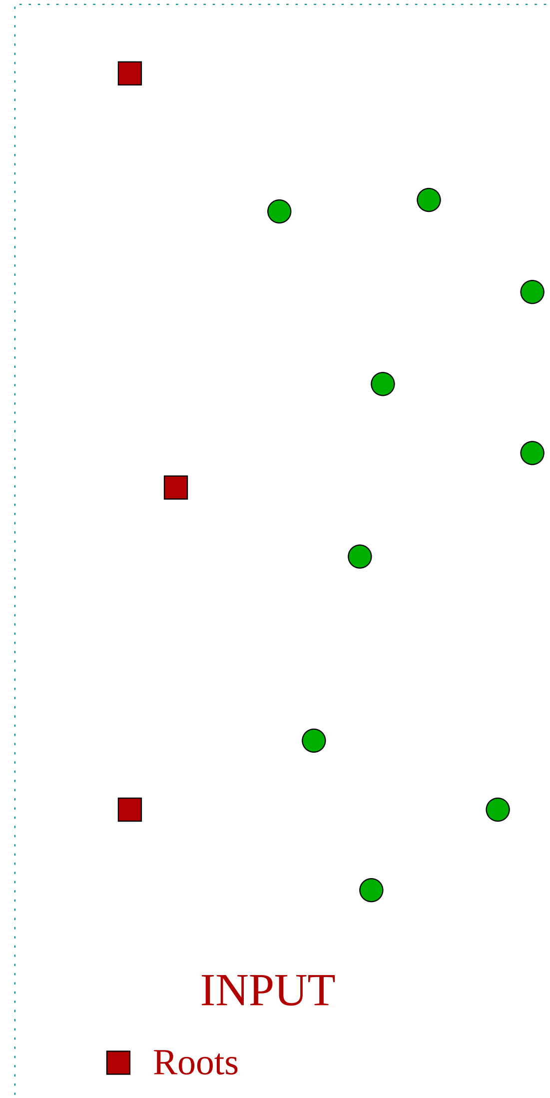

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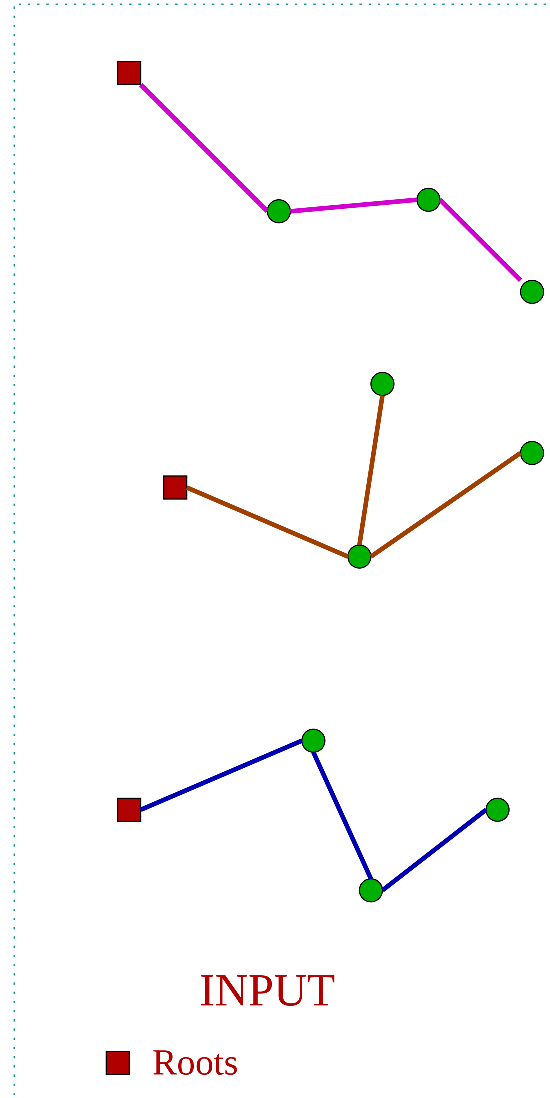

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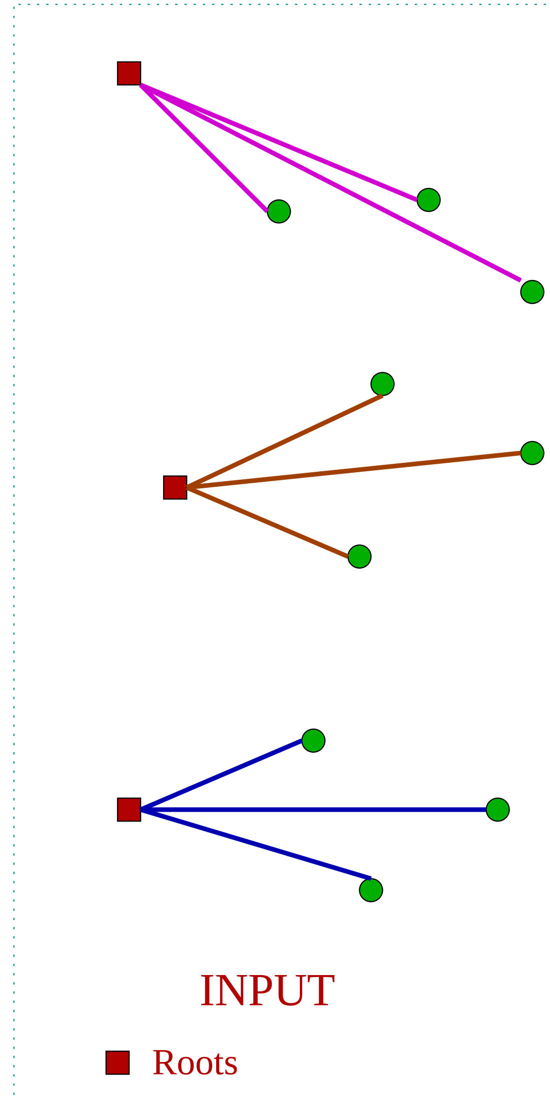
- Star cover: Cover with stars, same objective; may be rooted or unrooted.



Applications and literature

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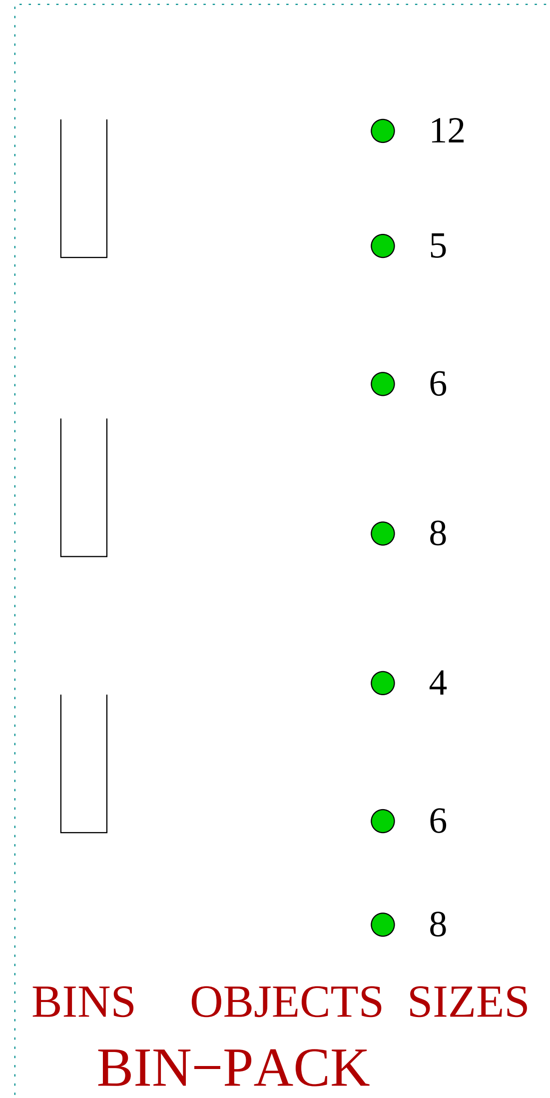
- Vehicle Routing: Vast amount of work, e.g. Survey [Toth, Vigo, 2002]
- Clustering is like covering with stars: Minimize maximum edge - k center [Dyer, Frieze, 1985], Minimize sum of edge lengths k median [Arya, et al 2001], Minimize sum of star radii [Charikar, Panigrahy, 2001].

Hardness (of rooted k-star cover)

- Reduction from BIN-PACK: Given elements U with sizes s_u, k bins of size B. Can we pack elements in k bins?



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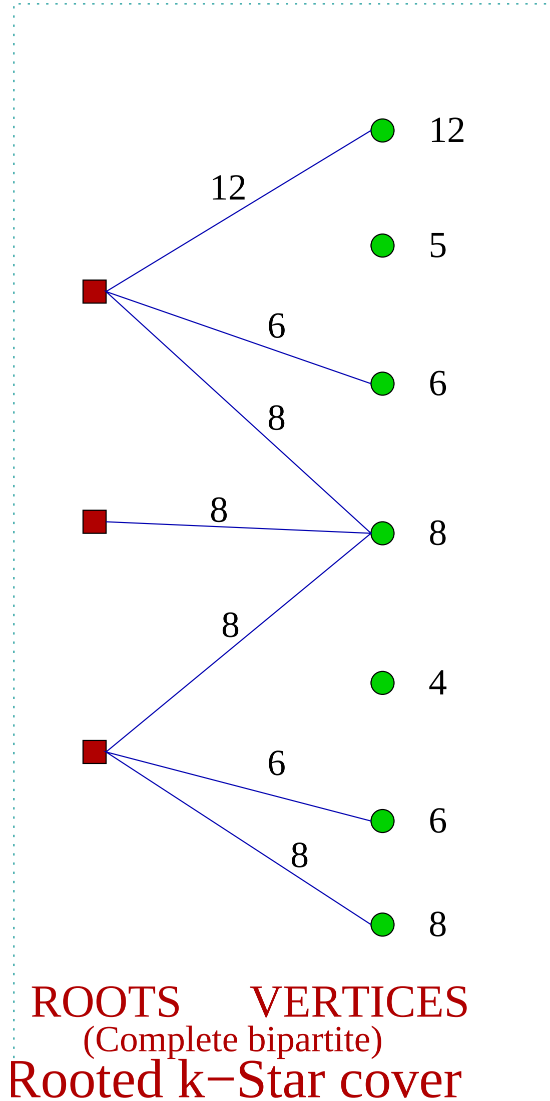

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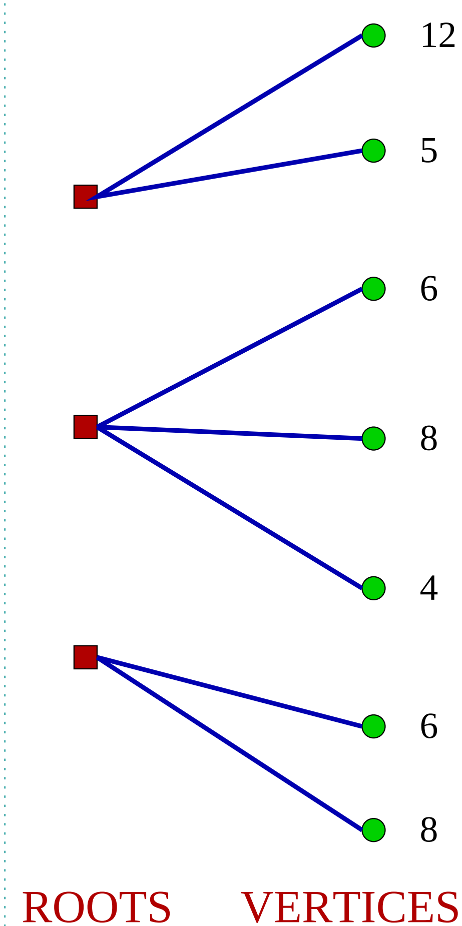

Solution

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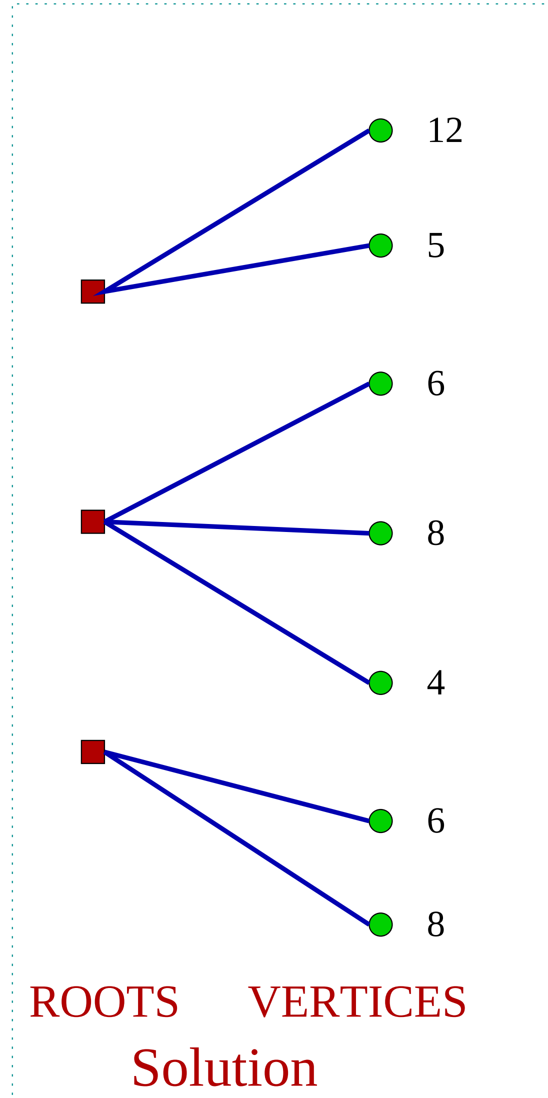
- Hardness of others follows by reducing to Rooted k-star cover.



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- Can be made strongly polynomial; approximation ratio worsens to $4 + \epsilon$.

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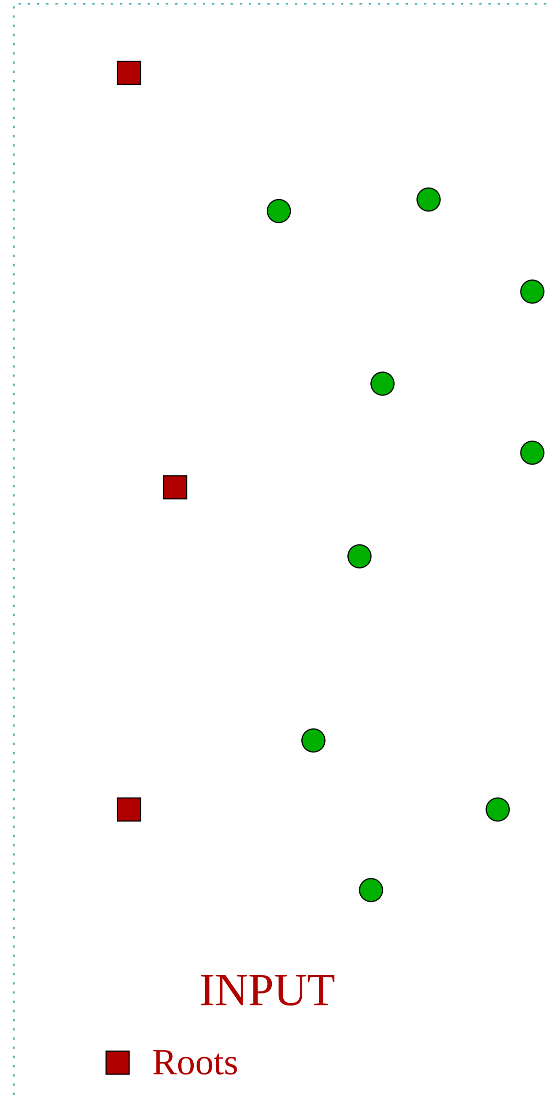
4. Match trees $\{S_i^j\}_i^j$ to roots in R within distance B from it.
   - If possible, return "success".
   - If impossible, return "fail".

Algorithm: Demonstration

1. Prune.



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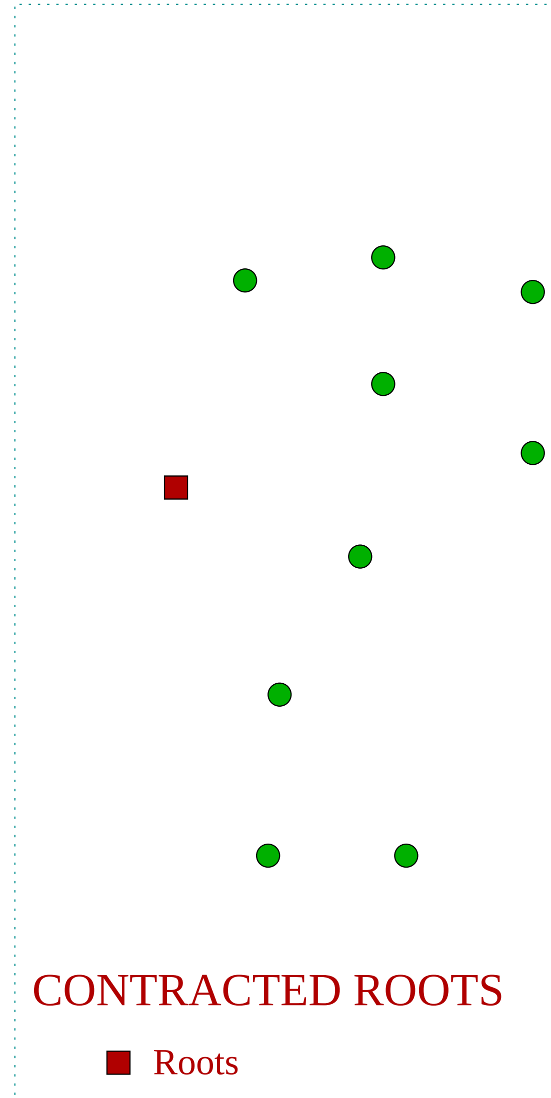

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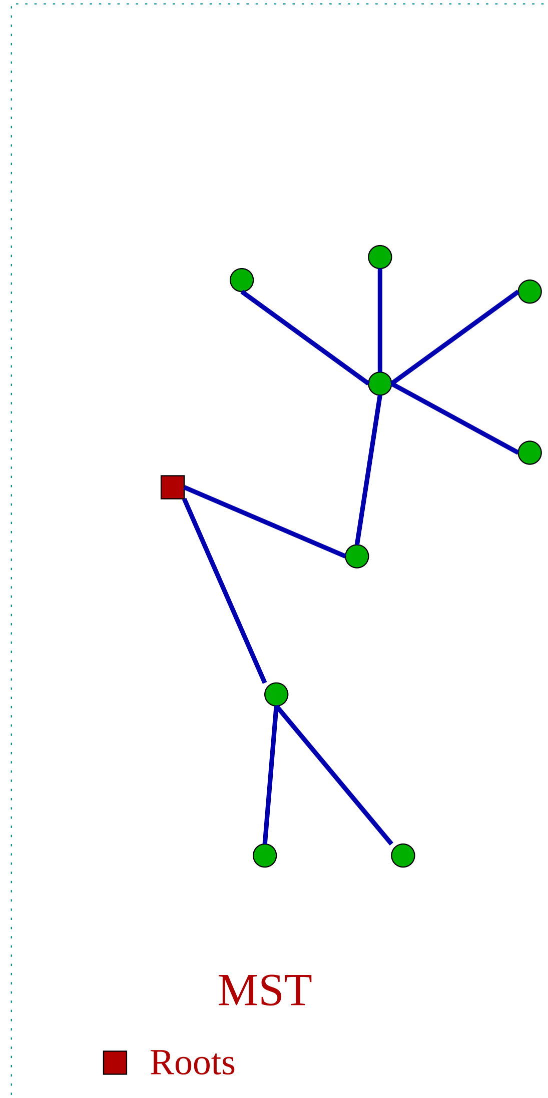

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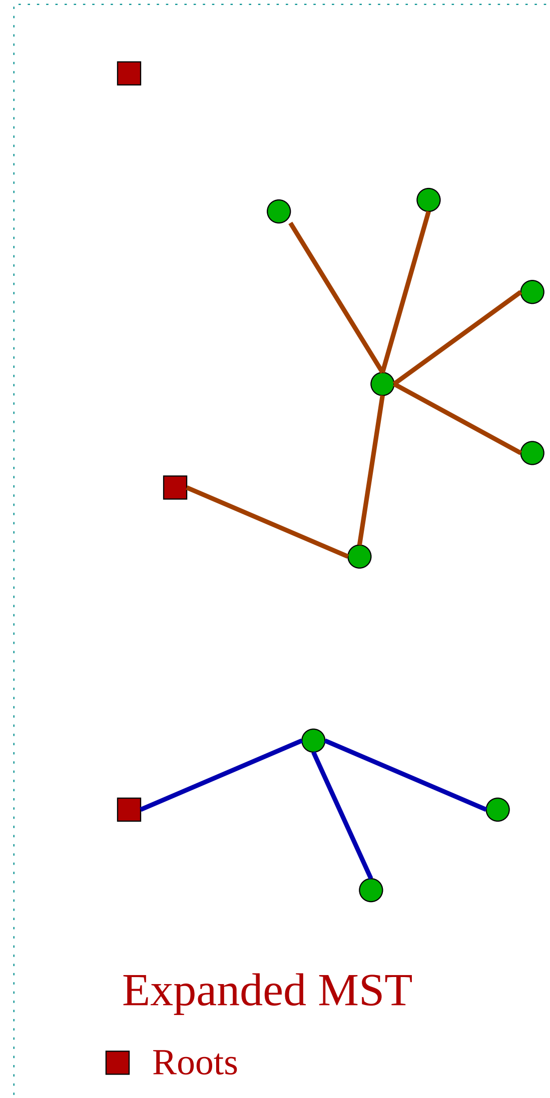

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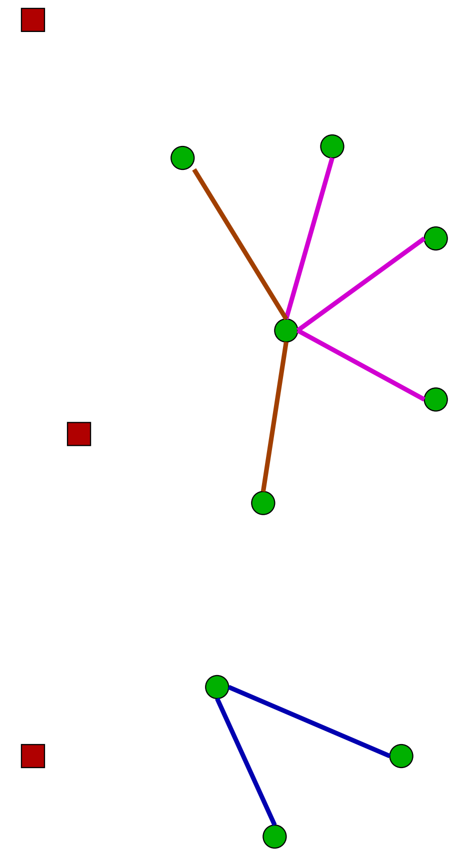

Decompose into light trees

Algorithm: Demonstration

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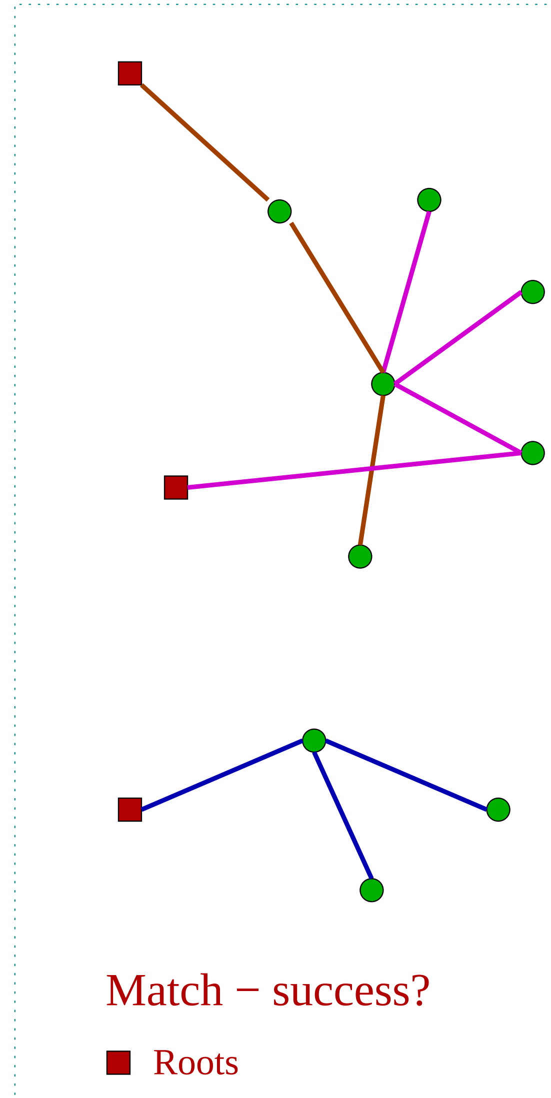

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$B^*|N(S)| \geq B^*|T^*(S)| \geq w(T^*(S)) \geq w(S) \geq B|S|$. $\square$

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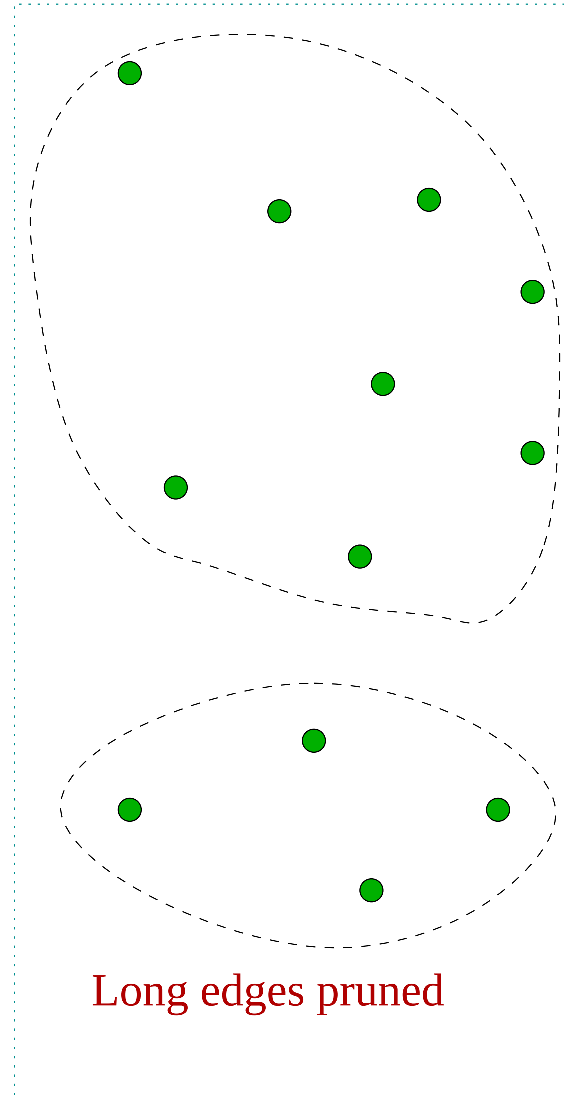

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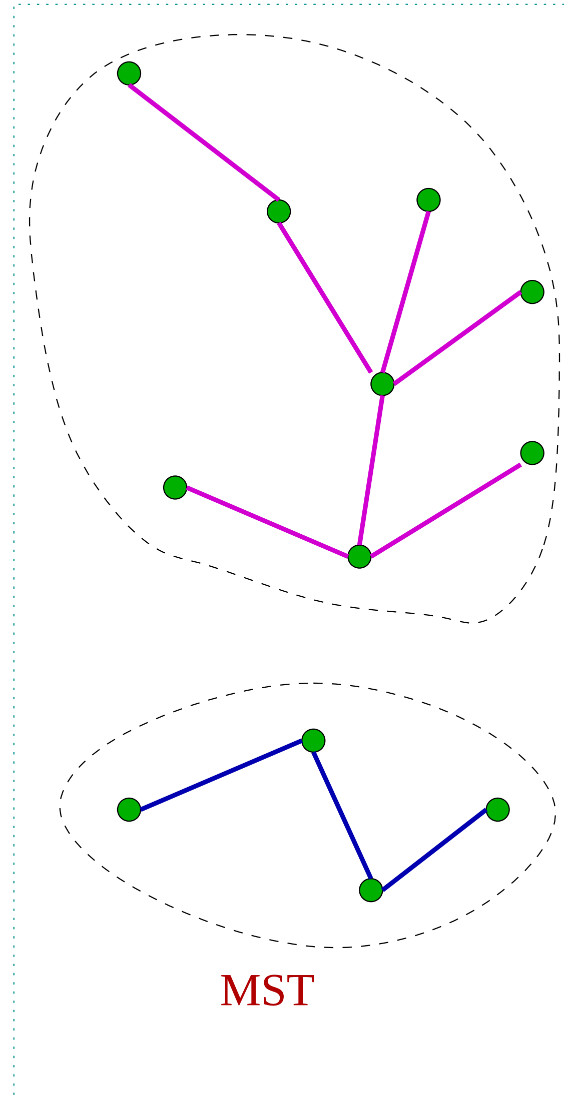

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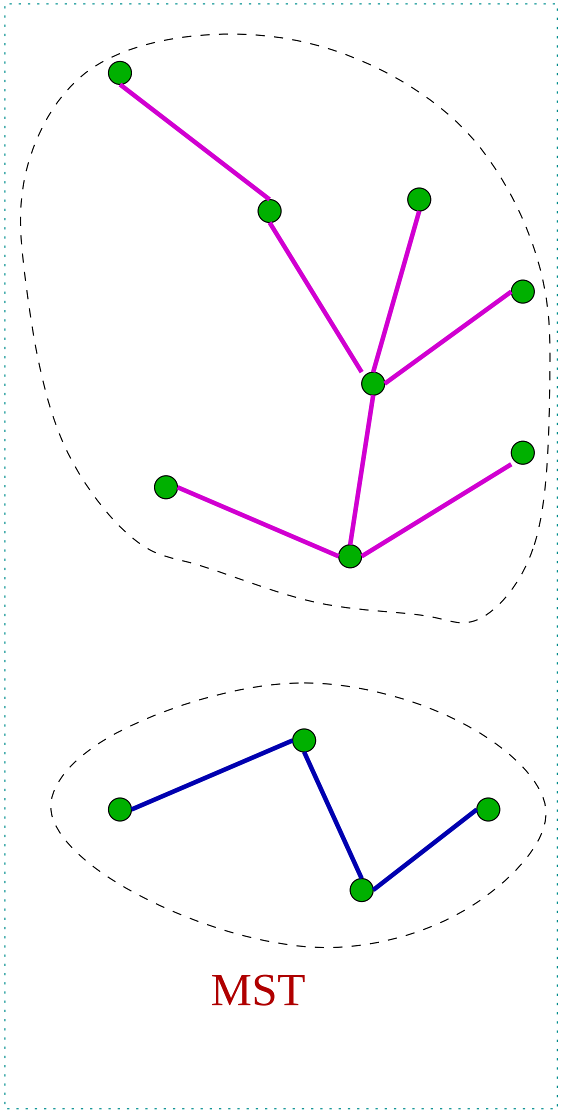

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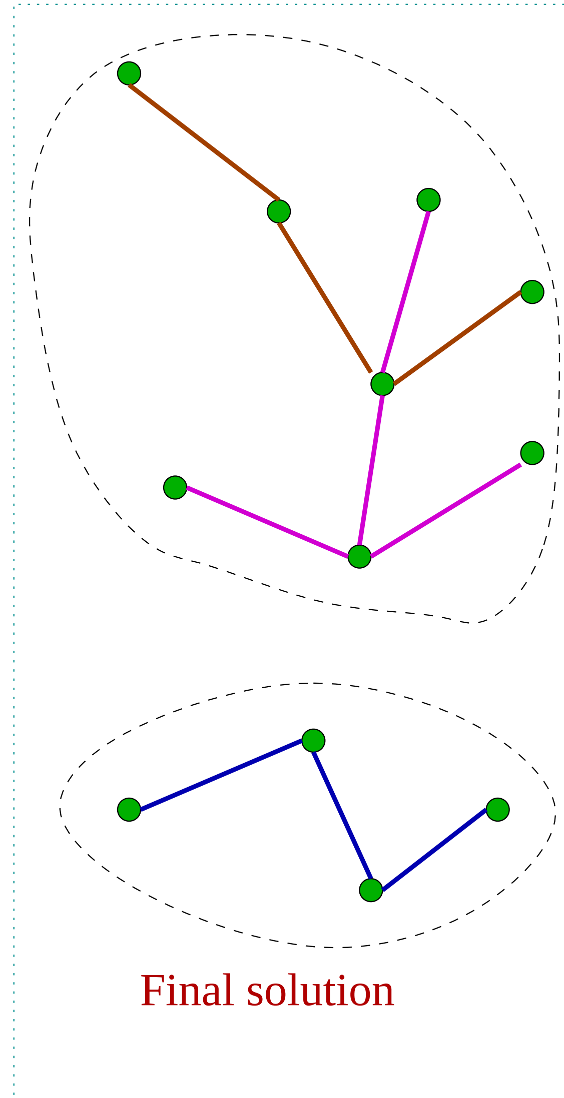

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Therefore $k_i^* \geq \frac{w(MST_i)}{2B} + \frac{1}{2} > k_i$. $\square$

Extensions and conclusion

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- Questions?