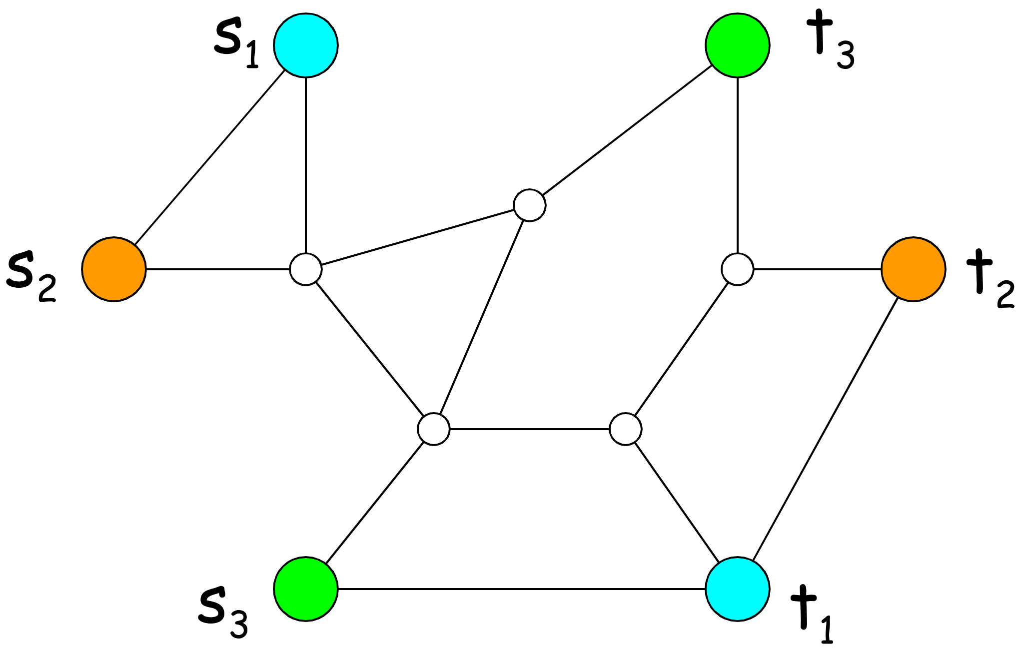
# A Network Connection Game

Elliot Anshelevich Anirban Dasgupta
Éva Tardos Tom Wexler

Cornell University

# Model

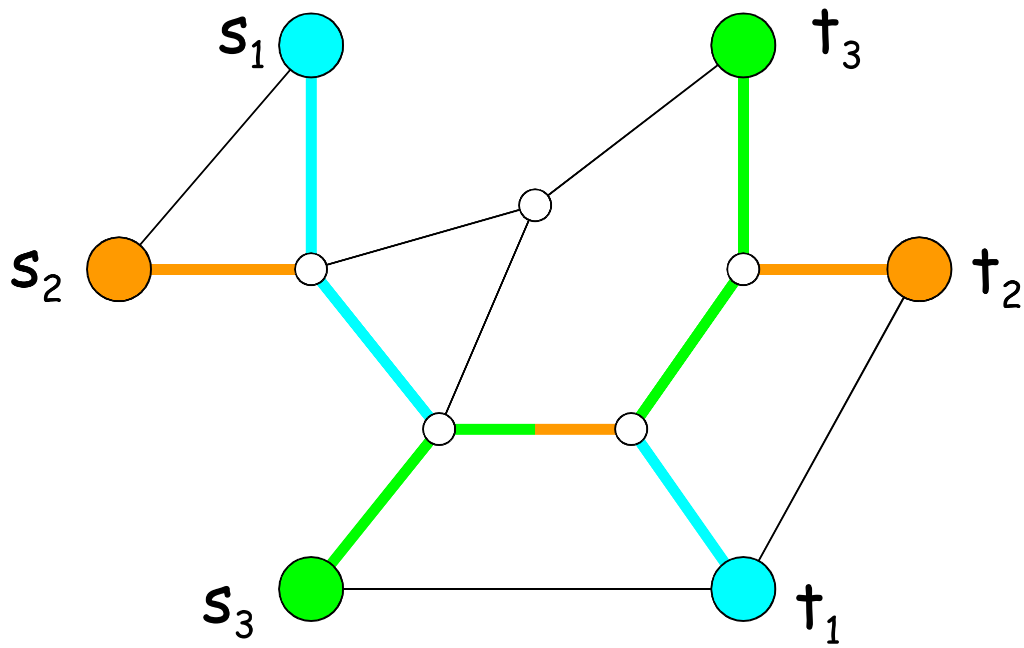



$G = (V,E)$ is an undirected graph with edge costs $c(e)$.

There are $k$ players.

Each player $i$ has a source $s_i$ and a sink $t_i$ he wants to have connected.

# Model (cont')

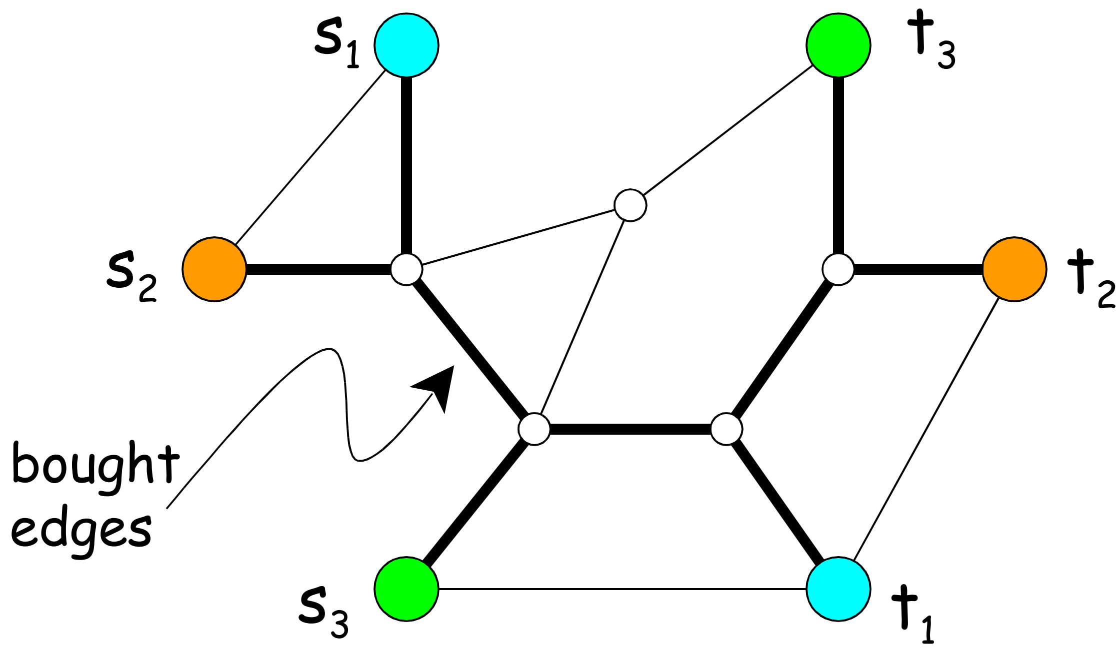



Player $i$ picks payment $p_i(e)$ for each edge $e$.

$e$ is bought if total payments $\geq c(e)$.

Note: any player can use bought edges

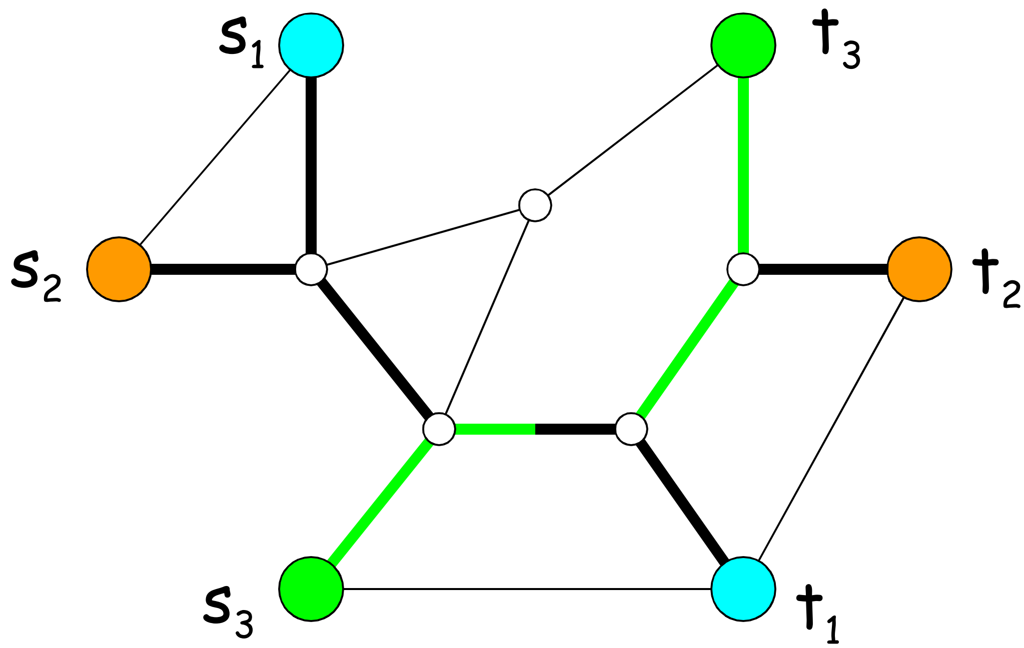
# The Game




Each player $i$ has only 2 concerns:

1) Must be a bought path from $s_i$ to $t_i$

# The Game

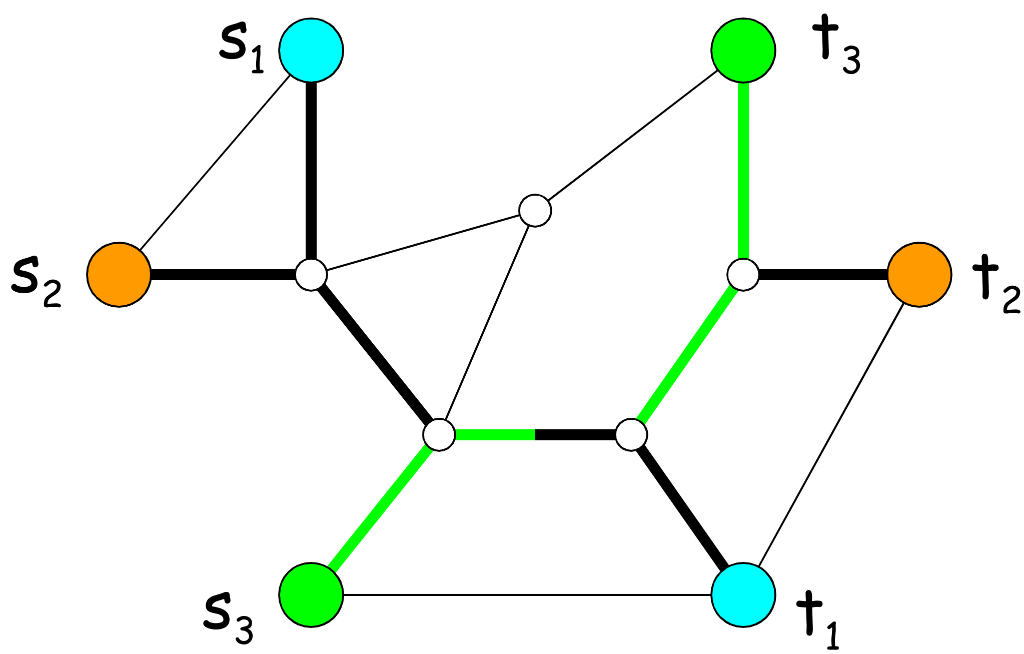



Each player $i$ has only 2 concerns:

1) Must be a bought path from $s_i$ to $t_i$

2) Given this requirement, $i$ wants to pays as little as possible.

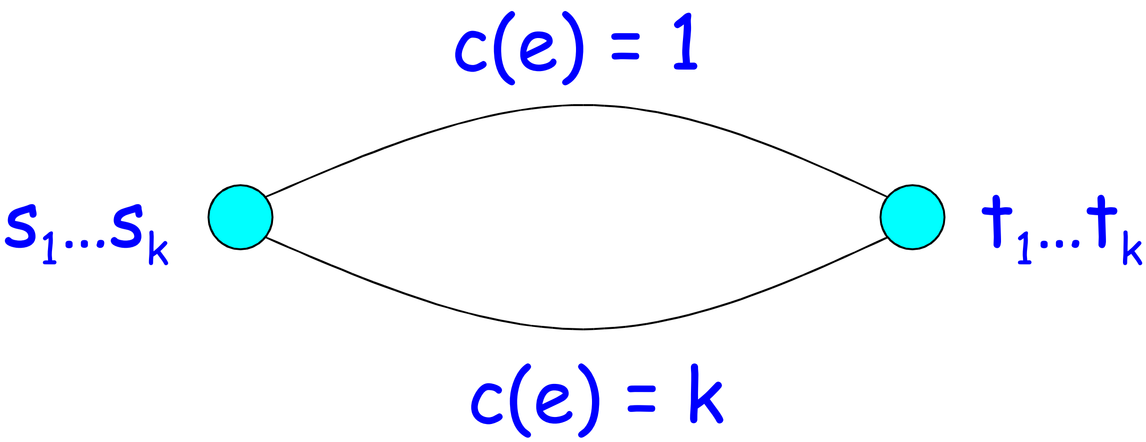
# Nash Equilibrium




A Nash Equilibium (NE) is set of payments for players such that no player wants to deviate.

Note: player i doesn't care whether other players connect.

# An Example




One NE:

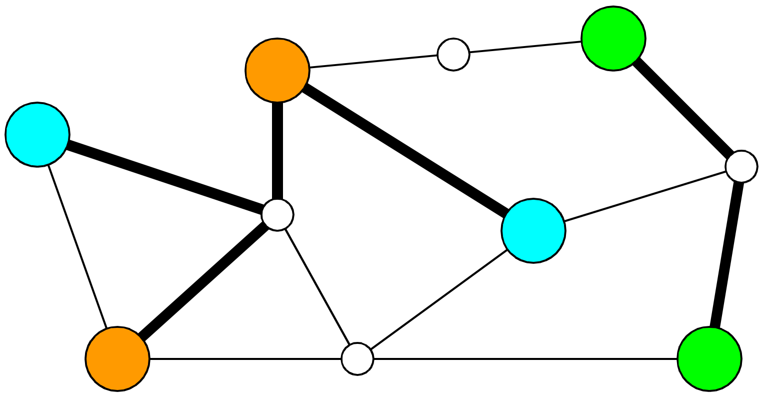
Each player pays $1/k$ to top edge.

Another NE:

Each player pays 1 to bottom edge.

Note: No notion of "fairness"; many NE that pay unevenly for the cheap edge.

# Three Observations

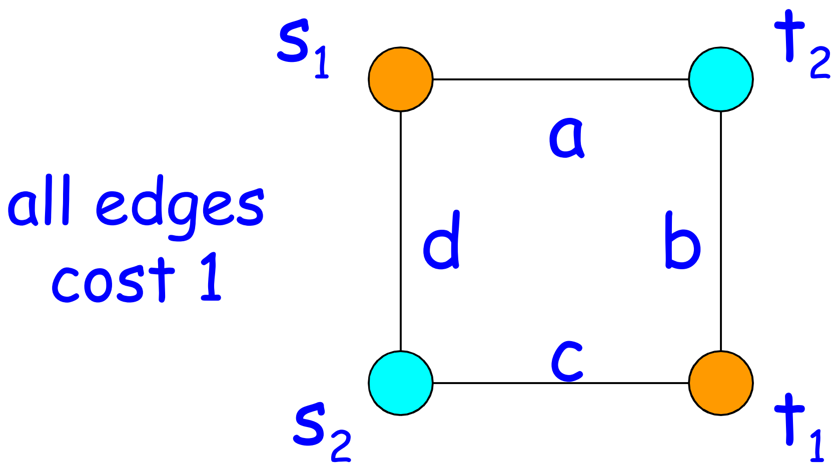


1) The bought edges in a NE form a forest.

2) Players only contribute to edges on their $s_i$-$t_i$ path in this forest.

3) The total payment for any edge $e$ is either $c(e)$ or $0$.

# Example 2: No Nash



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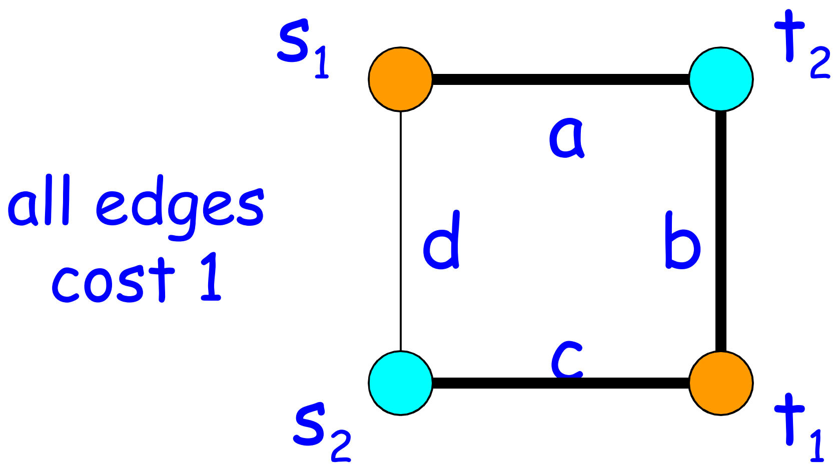


We know that any NE must be a tree:
WLOG assume the tree is a,b,c.

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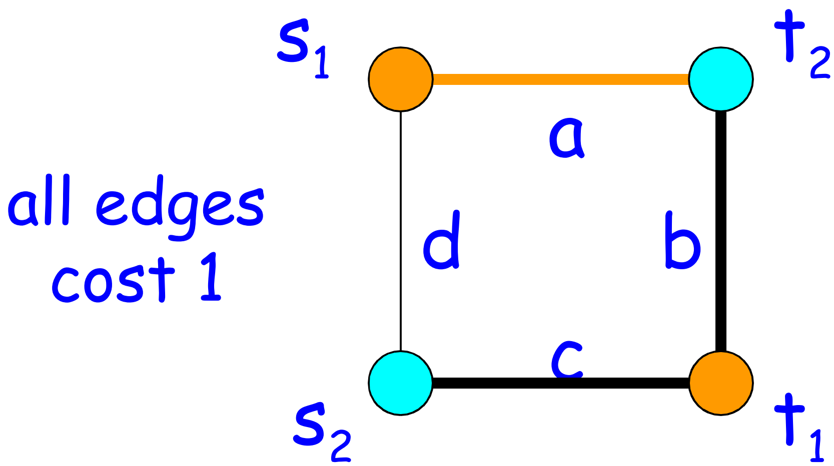


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ï Only player 1 can contribute to a.

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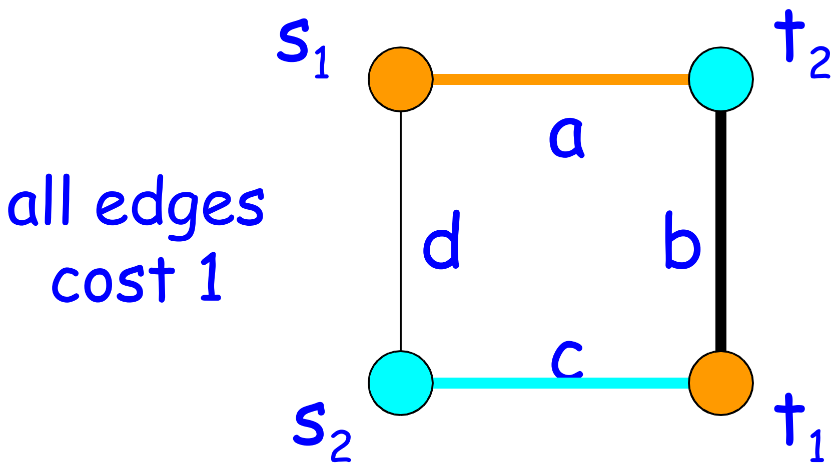


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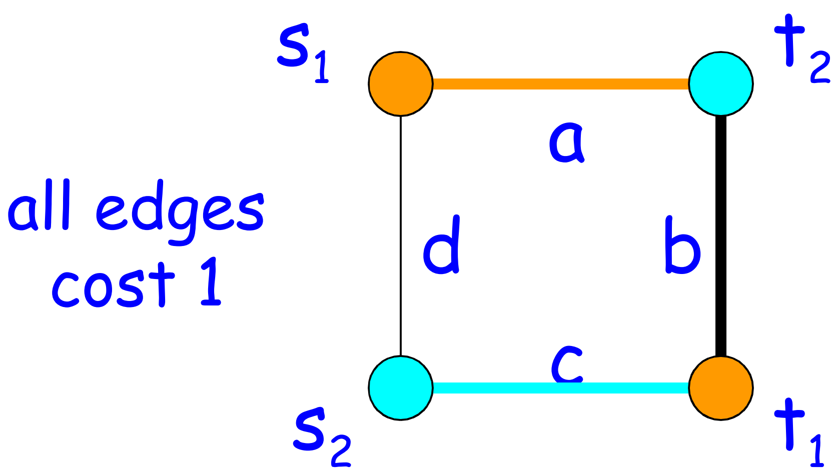



We know that any NE must be a tree: WLOG assume the tree is a,b,c.

ï Only player 1 can contribute to a.

ï Only player 2 can contribute to c.

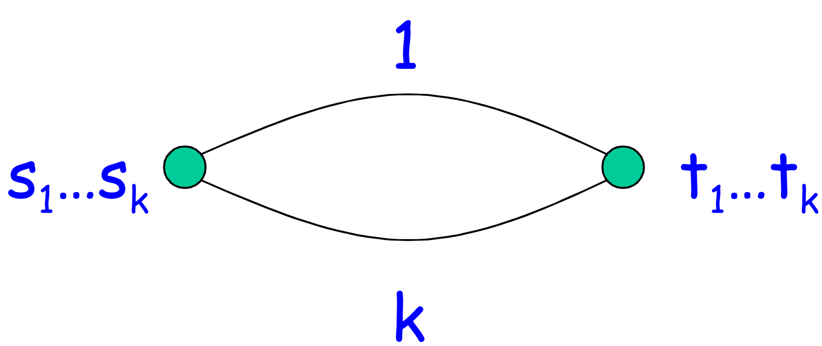
ï Neither player can contribute to b, since d is tempting deviation.

# Evaluating Outcomes: The Price of Anarchy

$$\text{Traditional P. of A.} = \frac{\text{cost(worst NE)}}{\text{cost(OPT)}}$$

[Papadimitriou]
[Roughgarden, Tardos]

(Min cost Steiner forest)




$$\text{Optimistic P. of A.} = \frac{\text{cost(best NE)}}{\text{cost(OPT)}}$$

# Related Work

Generalized Steiner tree

[Goemans, Williamson;…]

ï Centralized problem: connect pairs

Cost sharing [Jain, Vazirani;…]

ï Get players to pay for a tree

ï Players don't specify edge payment

Price of Anarchy [Papadimitriou; Koutsoupias, P; Roughgarden, Tardos;…]

Network creation game [Fabrikant, Luthra, Maneva, Papadimitriou, Shenker]

ï Players always purchase 1 edge

ï Players care about distances

# Outline

ï Introduction

ï Definitions

ï Two examples

ï Optimistic price of anarchy

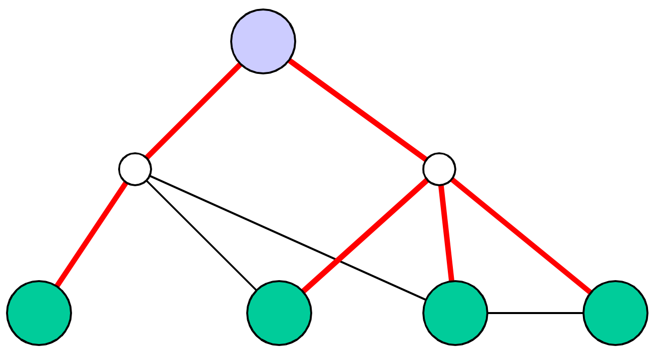
ï Single source games

ï General games (briefly)

ï A few extensions

# Single Source Games

($s_i = s$ for all $i$)



Thm: In any single source game, there is always a NE that buys OPT.

meaning 2 things…

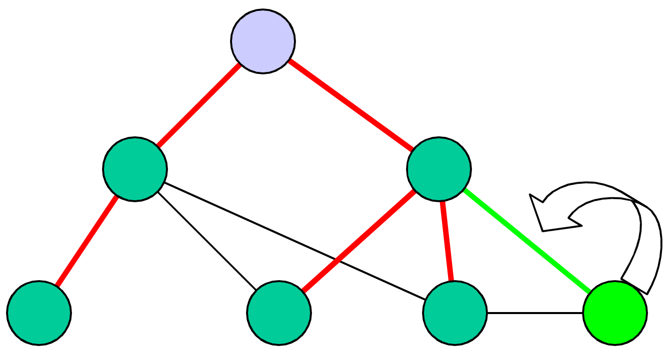
ï There is always a NE

ï The Price of Anarchy is 1!

Note: Existence result… we'll be able to extend this to an approximation algorithm.

# Simple Case: MST

It's easy if all nodes are terminals…



Players buy edge above them in OPT.

Claim: This is a Nash Equilibrium.

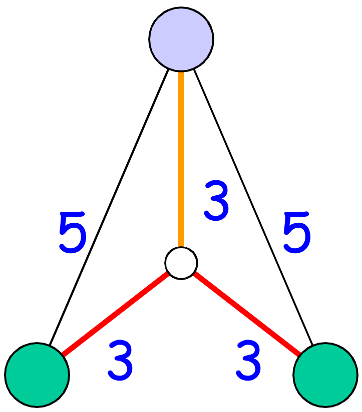
i unhappy => can build cheaper tree

Typically we will have Steiner nodes.

Who buys the edge above these?

# Attempting to Buy Edges

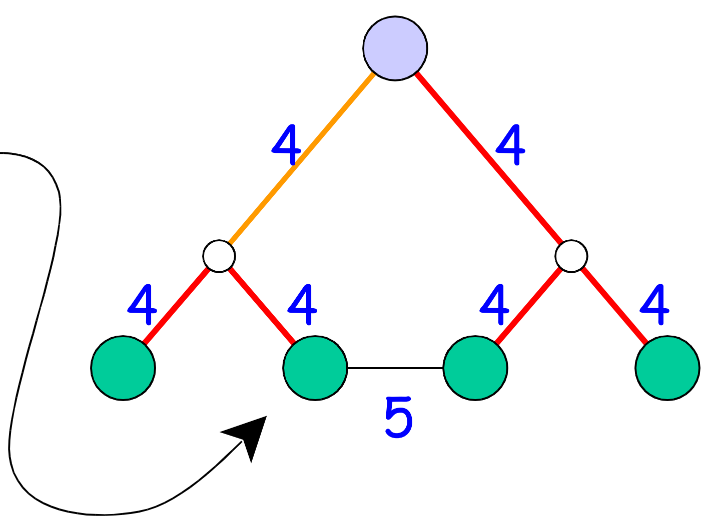
1) Can we get a single player to pay?




Both players must help buy top edge.

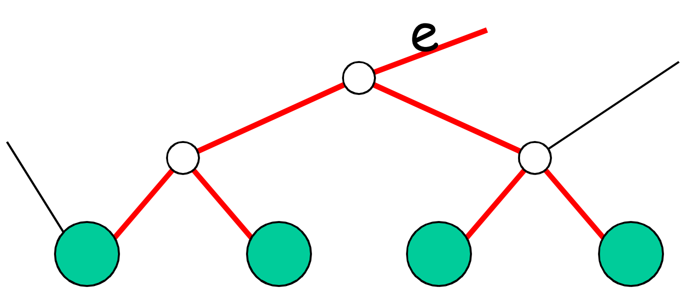
2) Can we split edge costs evenly?

Second node won't pay more than 5 in total.



# Idea for Algorithm

In both examples, players were limited by possible deviations.




Pay for edges in OPT from the bottom up, greedily, as constrained by deviations.

If we buy all edges, we're done!

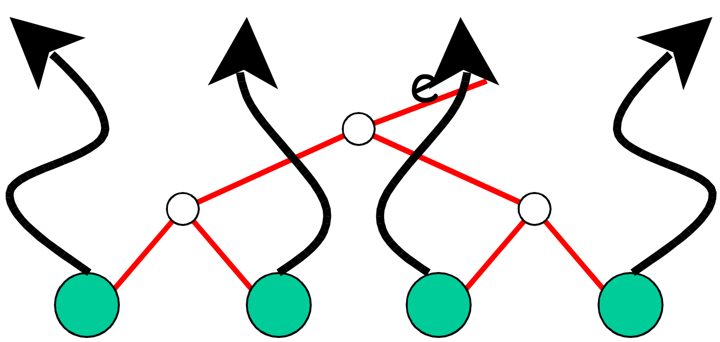
# Idea for Proof

If greedy doesn't pay for e, we'll try to show that the tree is not OPT.

ï All players have poss. deviations.

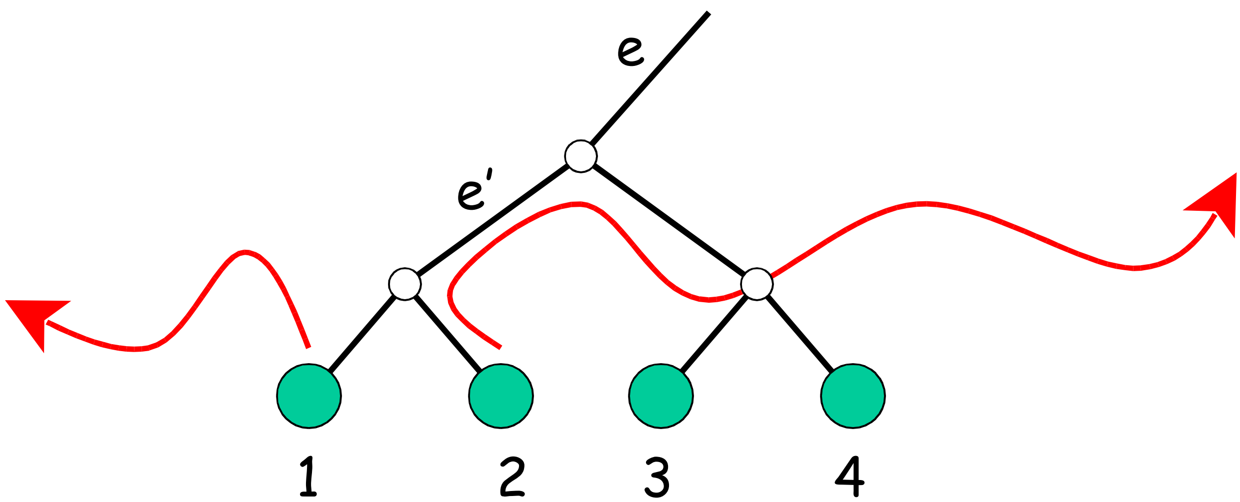
ï Deviations and current payments must be equal.

ï If all players deviate, all connect, but pay less.



# A Possible Pitfall

Suppose greedy alg. can't pay for e.



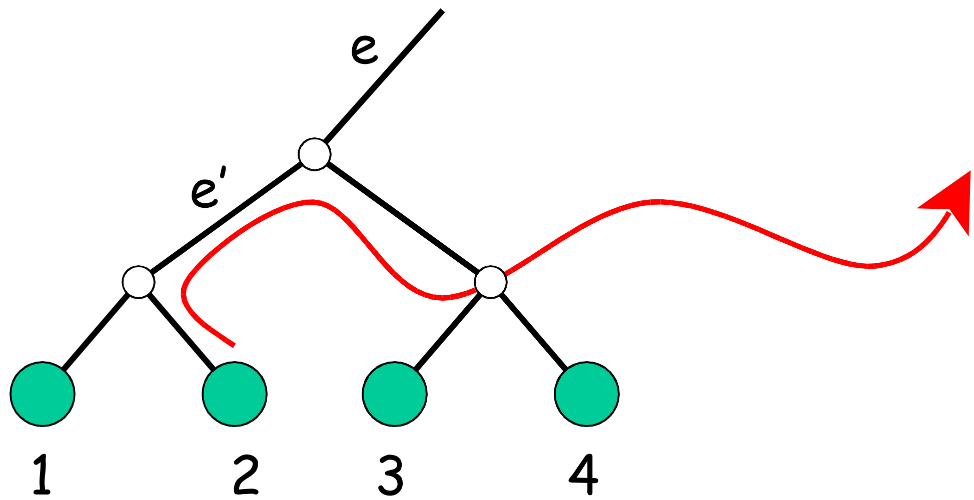

Further, suppose 1 & 2 share cost(e')

Consider 1 & 2 both deviating…

Player 1 stops contributing to e'

Danger: 2 still needs this edge!

# Safely Selecting Paths

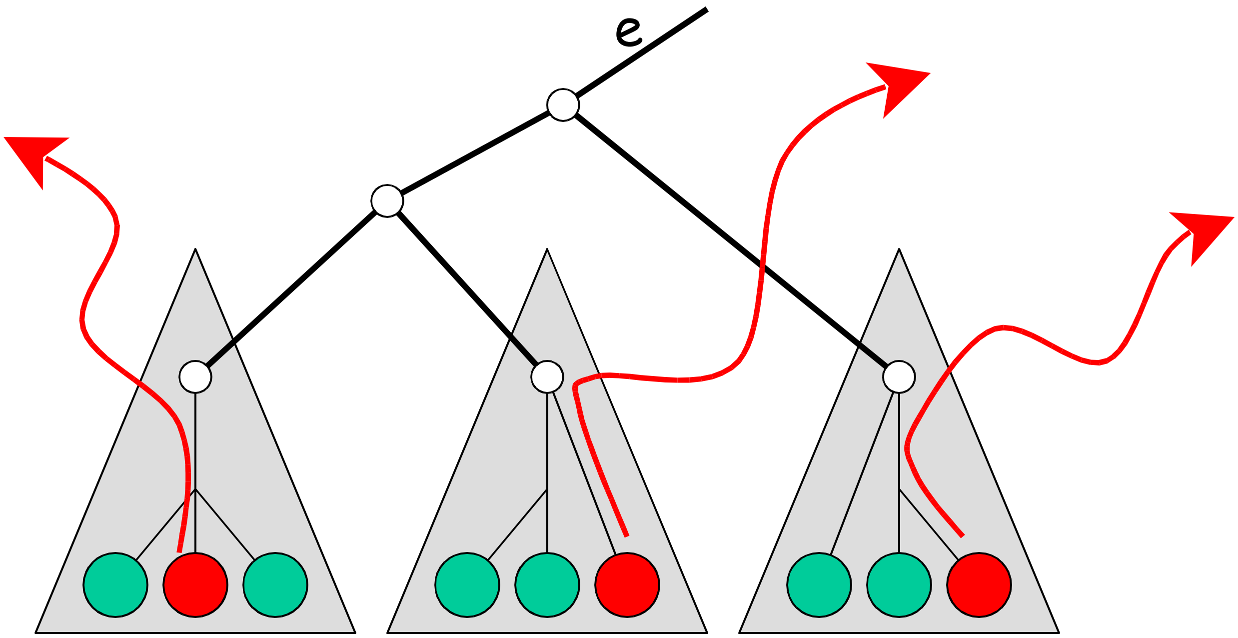



Shouldn't allow player 1 to deviate.

If only 2 deviates, all players reach the source.

Idea: should use the "highest" deviating paths first.

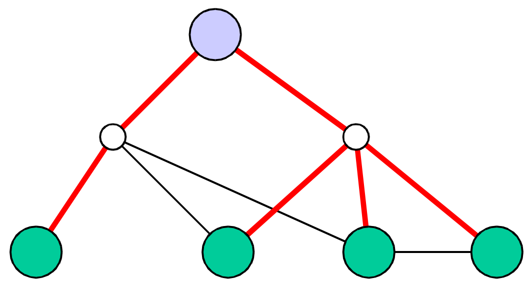
# Safely Selecting Paths (cont')




We may have to select multiple alternate paths.

Remember: Not trying to find a NE, just a contradiction.

# Recap



If greedy doesn't pay for some edge e, we can make cheaper tree.

This is a contradiction!

Therefore, the algorithm:

Greedily buy edges from bottom up

finds a NE that buys OPT.

...but we may not have OPT on hand...

# Single Source in Polytime

Thm: For single source, can find a $(1+\varepsilon)$-approx. NE in polytime on an $\alpha$-approx. Steiner tree.

- $\alpha$ = best Steiner tree approx. (1.55)
- $\varepsilon > 0$, running time depends on $\varepsilon$.

Pf Sketch: Alg. basically the same…

Since tree $\alpha$-approx, might not be able to pay for all of an edge.

If we can't buy $> \varepsilon$ of an edge, use deviations to build a cheaper tree.

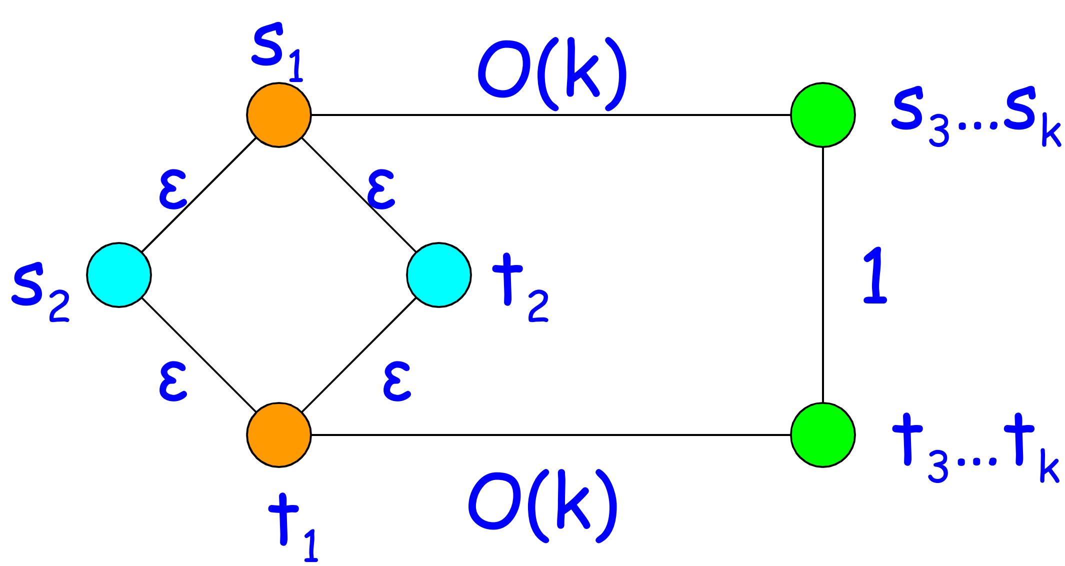
# Outline

- Introduction
- Definitions
- Two examples
- Optimistic price of anarchy
- Single source games
- General games (briefly)
- A few extensions

# General Games (An Example of High PoA)

Saw a game on a 4-cycle with no NE.

If NE exist, is the best NE cheap?




OPT costs ~1, but it's not a NE.

The only NE costs $O(k)$, so optimistic price of anarchy is almost $k$.

# Result for General Games

We know we might not have any NE, so we're going to have to settle for approximate NE.

How bad an approximation must we have if we insist on buying OPT?

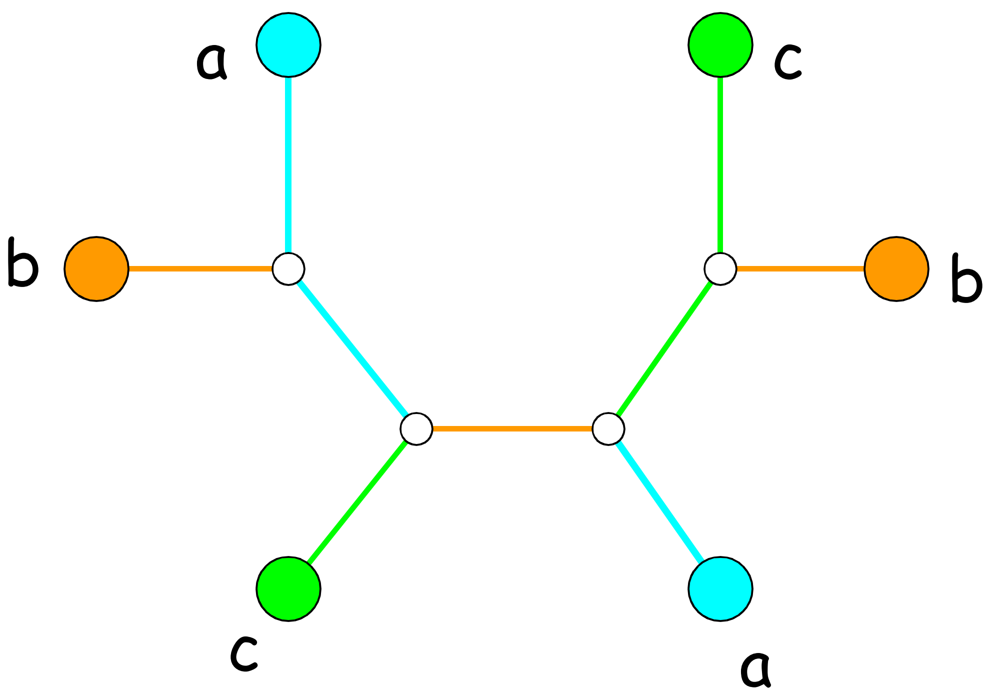
Thm: For any game, there exists a 3-approx. NE. that buys OPT.

# Proof Idea

Break tree up into chunks.

Use optimality of tree to show that any player buying a single chunk has no incentive to deviate.

Ensure that every chunk is paid for, and each player gets at most 3.



# Extensions

Thm: For any game, we can find a ($3 + \varepsilon$)-approx. NE on a 2-approx to OPT in polytime.

Result generalizes to game where player $i$ has > 2 terminals to connect.

Results for single source game extend to directed graphs.

All results can handle addition of $max(i)$, a price beyond which player $i$ would rather not connect at all.