A Bézier-Based Approach to Unstructured Moving Meshes

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Motivation

• Meshing
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- Sangria Project - Develop simulation techniques for RBC flow.
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• Moving Meshes to Track Boundaries Simplify Numerical Schemes.
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• Sangria Project - Develop simulation techniques for RBC flow.
• Moving Meshes to Track Boundaries
  Simplify Numerical Schemes.
• Curved Meshes for more accurate displacements.
Bézier Curves

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  \[
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- Quadratic Bézier Curves
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  B(t) = (1 - t)^2 p_0 + 2t(1 - t)p_1 + t^2 p_2
  \]
Bézier Curves (Picture)

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Why use Bézier Curves?

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- Variation Diminishing property
Bézier Triangles

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• Importance of Control Net

• Analogues in Higher Dimension (Bézier Tetrahedra)
Mesh Hierarchy

- Curved Mesh

![Curved Mesh Diagram](image)
Mesh Hierarchy

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- Control Mesh
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- Logical Mesh
Mesh Quality

- What is Mesh Quality?
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• Mesh Size and Mesh Grading are ’macro-quality’
Bézier Triangle Quality

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- Maintain Quality Control Mesh
- First things first, second things second.
Mesh Cleaning

• Given a mesh of poor quality
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- Given a mesh of poor quality
- Given a sizing function
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- Coarsen the Mesh to assure keep output size low
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• *Localized* Operations
• Operations that generalize well to 3-D
Edge Flips

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- Edge flips can be used as an atomic topological operation for many algorithms.
- Use edge flips to make the logical mesh Delaunay.
Bézier Mesh Coarsening

• Use traditional linear mesh coarsening algorithms. Identify a set of points to be removed.
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- Use Deviller’s algorithm for incremental vertex removal.
Bézier Mesh Refinement

- Identify poorly sized triangles.
Bézier Mesh Refinement

• Identify poorly sized triangles.
• Identify poor logical triangles.
Bézier Mesh Refinement

• Identify poorly sized triangles.
• Identify poor logical triangles.
• Use Ruppert Refinement to insert the circumcenters of logical triangles.
Curve Smoothing

• Identify 'overly curved' triangles using special metrics and smooth each edge.
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- Many metrics can be very easily computed for Bézier triangles.
- Identify star of control point in the control mesh.
- Use local linear mesh improvement algorithms to determine a new position.
Simulation Process

• Push Bézier mesh forward using some displacement field.
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- Push Bézier mesh forward using some displacement field.
- Use Edge flips to enforce Delaunay property on the logical mesh.
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Simulation Process

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- Solve Equations for next timestep. Rinse. Repeat.
Demos

- Pure Convection
- Convection Diffusion
- Navier-Stokes
Recap

- Use Bézier curves and triangles as a basis for curved meshes
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• Maintain linear quality and higher-order quality of curved elements
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• Use Bézier curves and triangles as a basis for curved meshes
• Maintain linear quality and higher-order quality of curved elements
• Use extensions of known linear algorithms to ensure ‘macro-quality’ of the mesh
Future Considerations

- 3-D. Most of this generalizes to higher dimension.
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• Implement More Exotic Elements.
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- Questions?