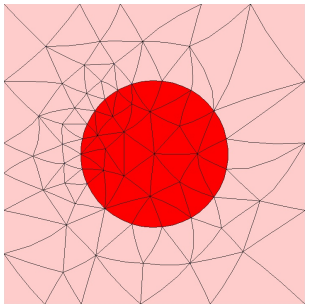
A Bézier-Based Approach to Unstructured Moving Meshes

Todd Phillips

David Cardoze

Gary Miller

Motivation

- Meshing 

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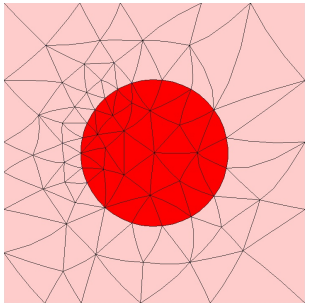

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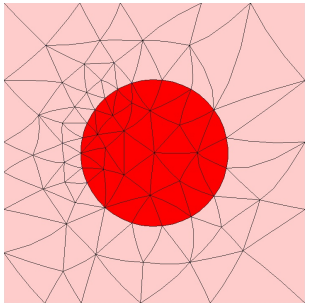

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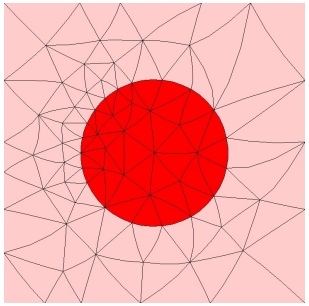

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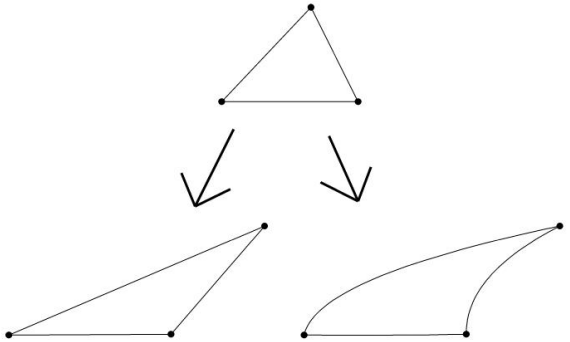

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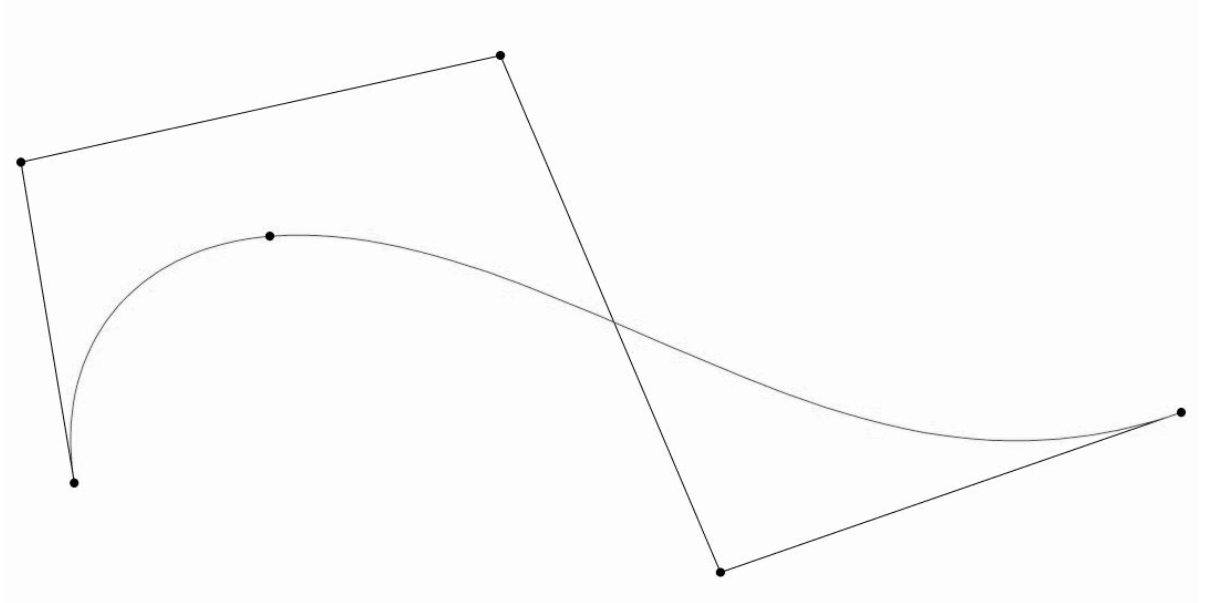
$$B(t) = (1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2$$

Bézier Curves (Picture)

- General Bézier Curves



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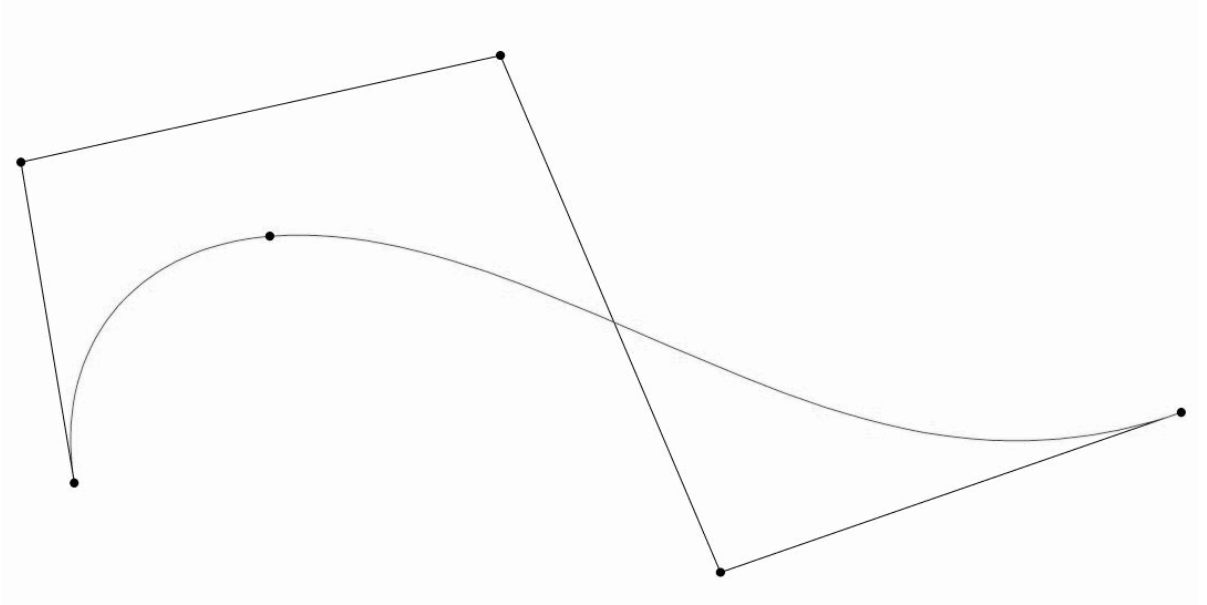

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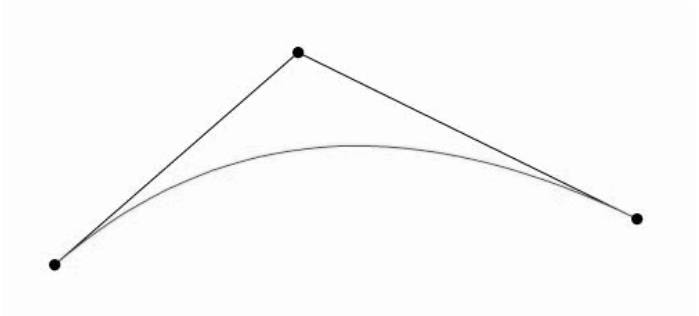

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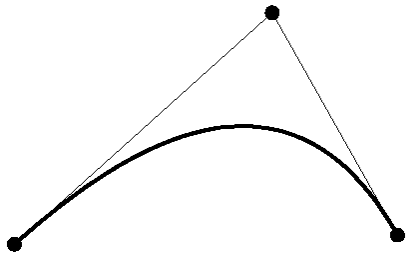

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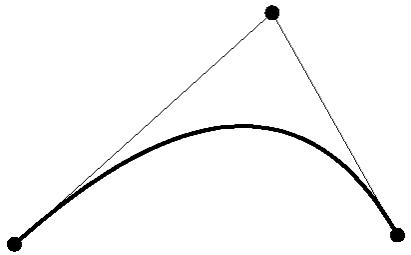

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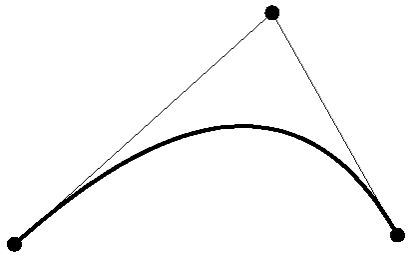

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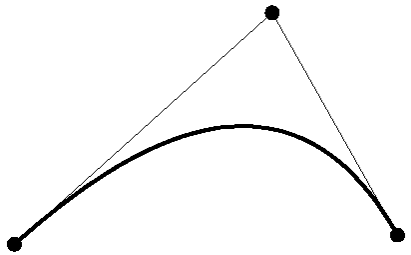

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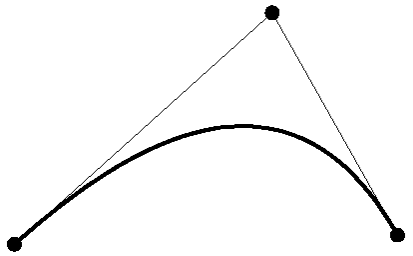

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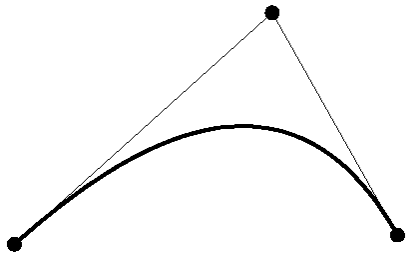

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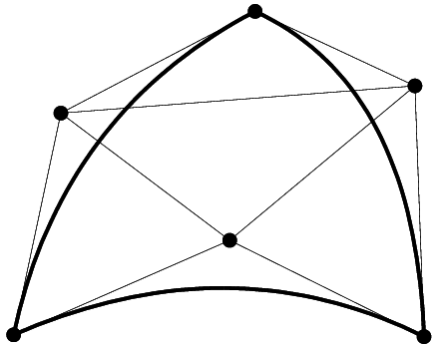

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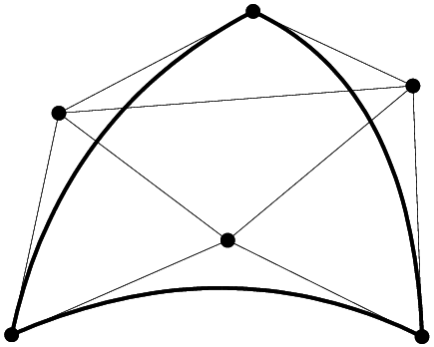

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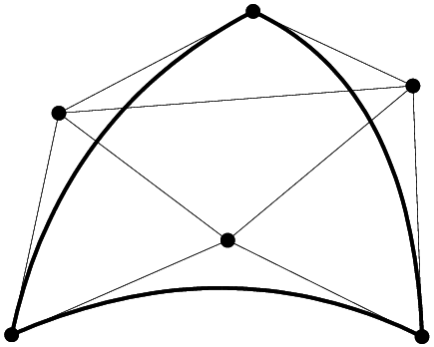
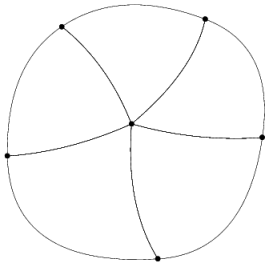


- Importance of Control Net
- Analogues in Higher Dimension (Bézier Tetrahedra)

Mesh Hierarchy

- Curved Mesh



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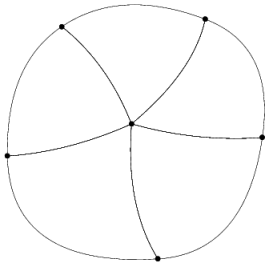

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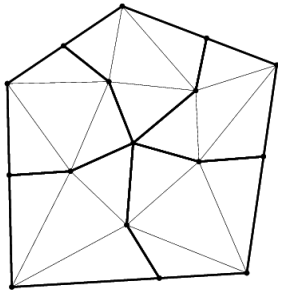

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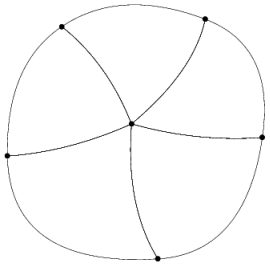

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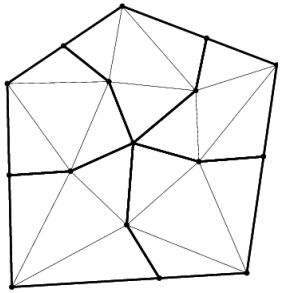
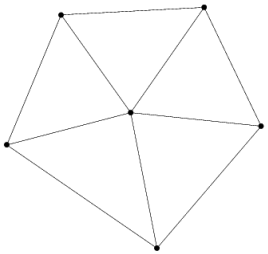


- Logical Mesh



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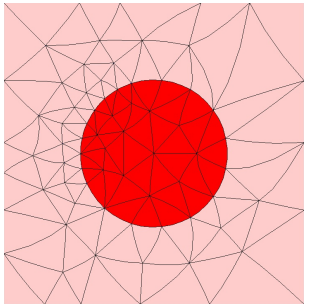

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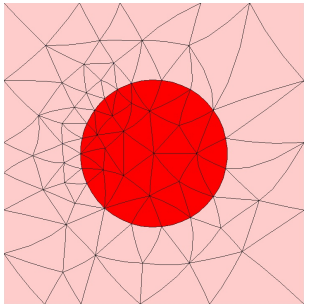

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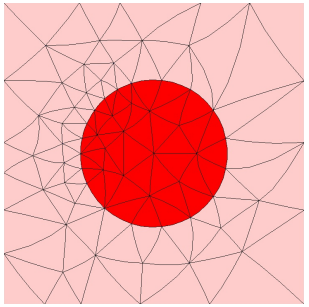

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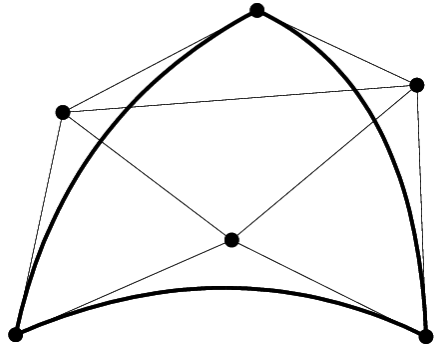

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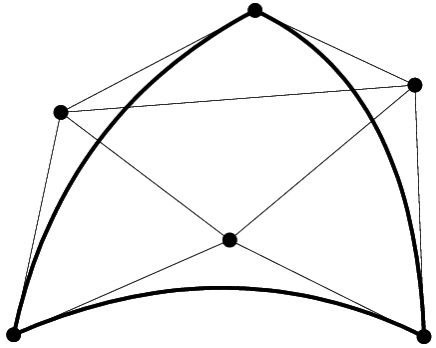

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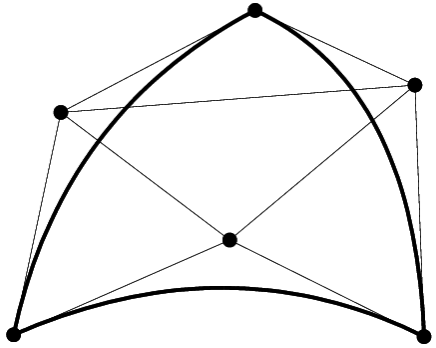

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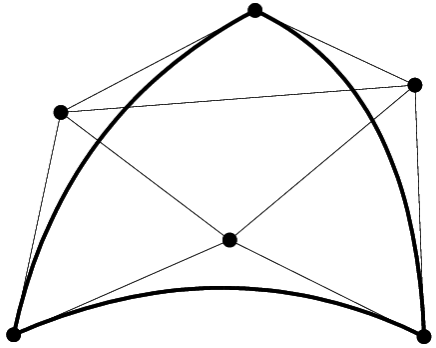

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- First things first, second things second.

Mesh Cleaning

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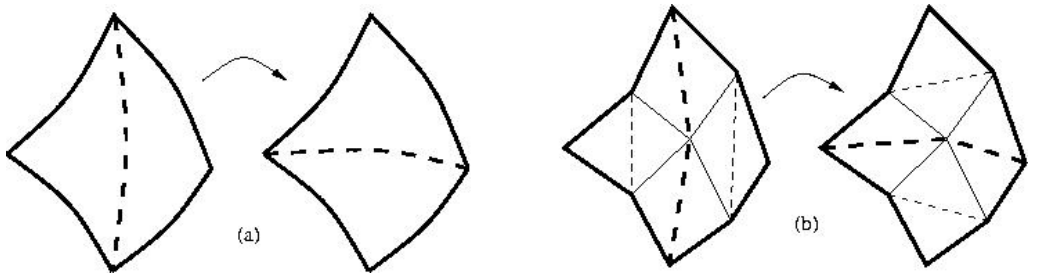
- Operations that generalize well to 3-D

Edge Flips



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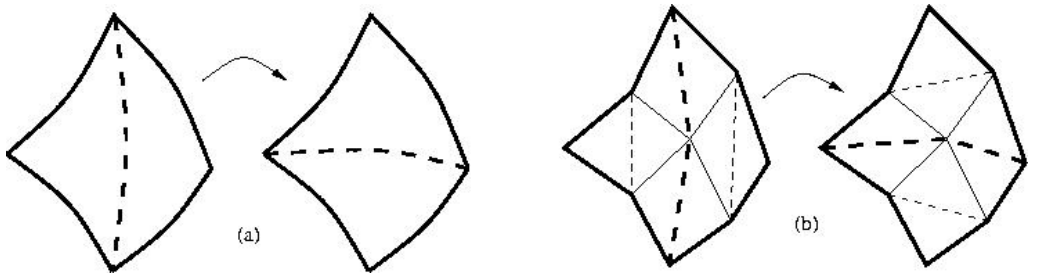

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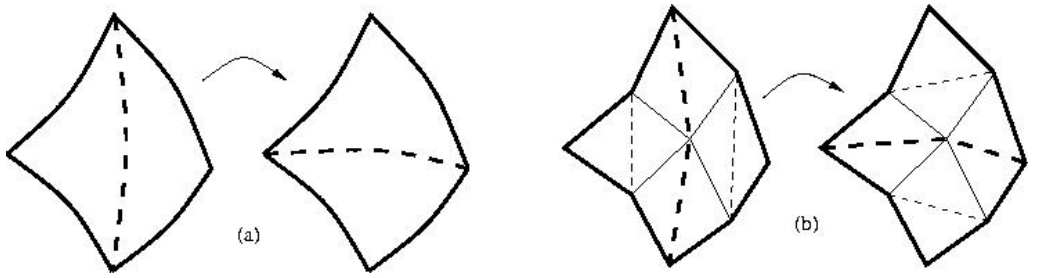

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- Use edge flips to make the *logical* mesh Delaunay.

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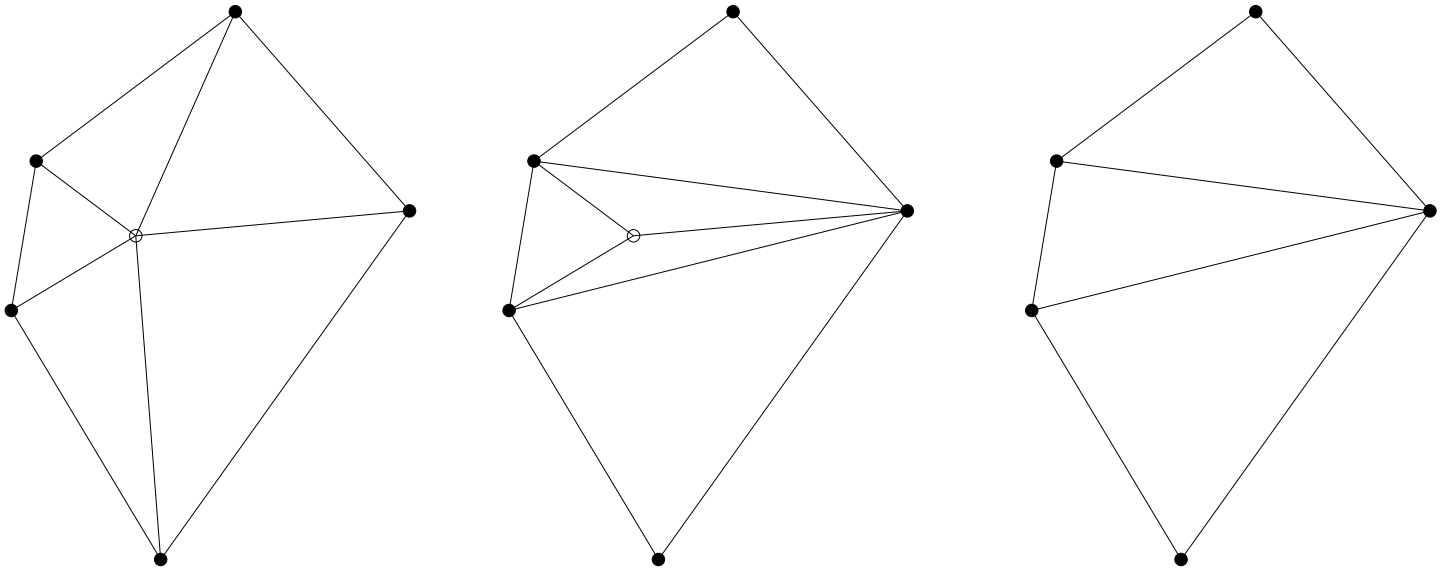
- Use Deviller's algorithm for incremental vertex removal.



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- Use Ruppert Refinement to insert the circumcenters of logical triangles.

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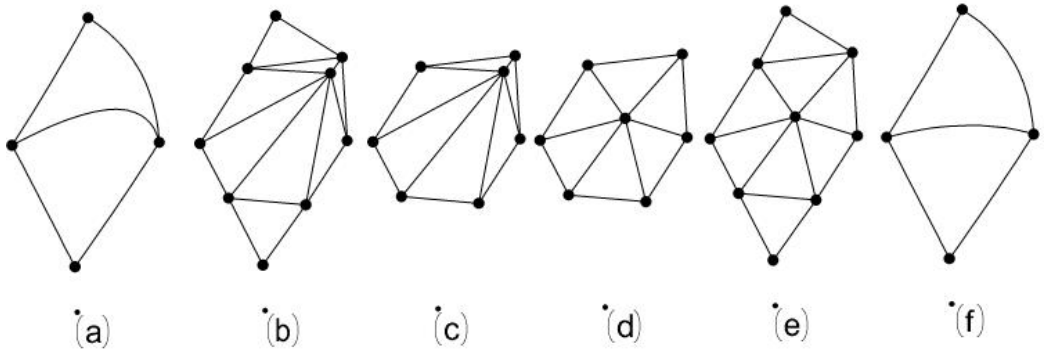

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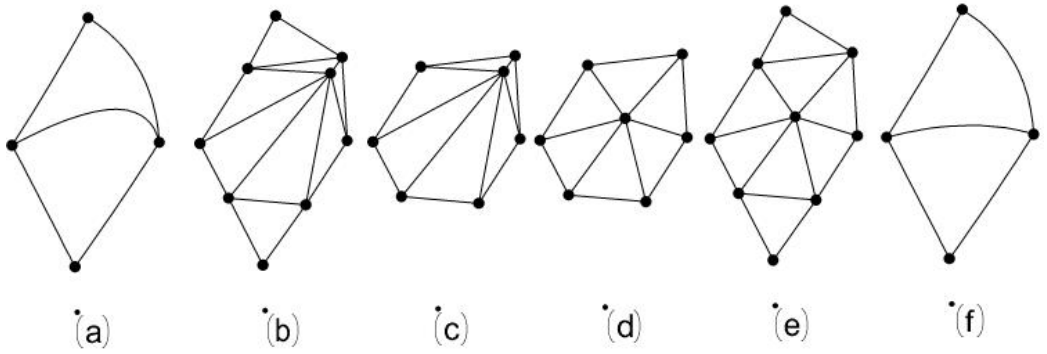

- Use local linear mesh improvement algorithms to determine a new position.

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- Solve Equations for next timestep. Rinse. Repeat.

Demos

- Pure Convection
- Convection Diffusion
- Navier-Stokes

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- Maintain linear quality and higher-order quality of curved elements
- Use extensions of known linear algorithms to ensure ’macro-quality’ of the mesh

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