On Nash Equilibria for a Network Creation Game

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Motivation

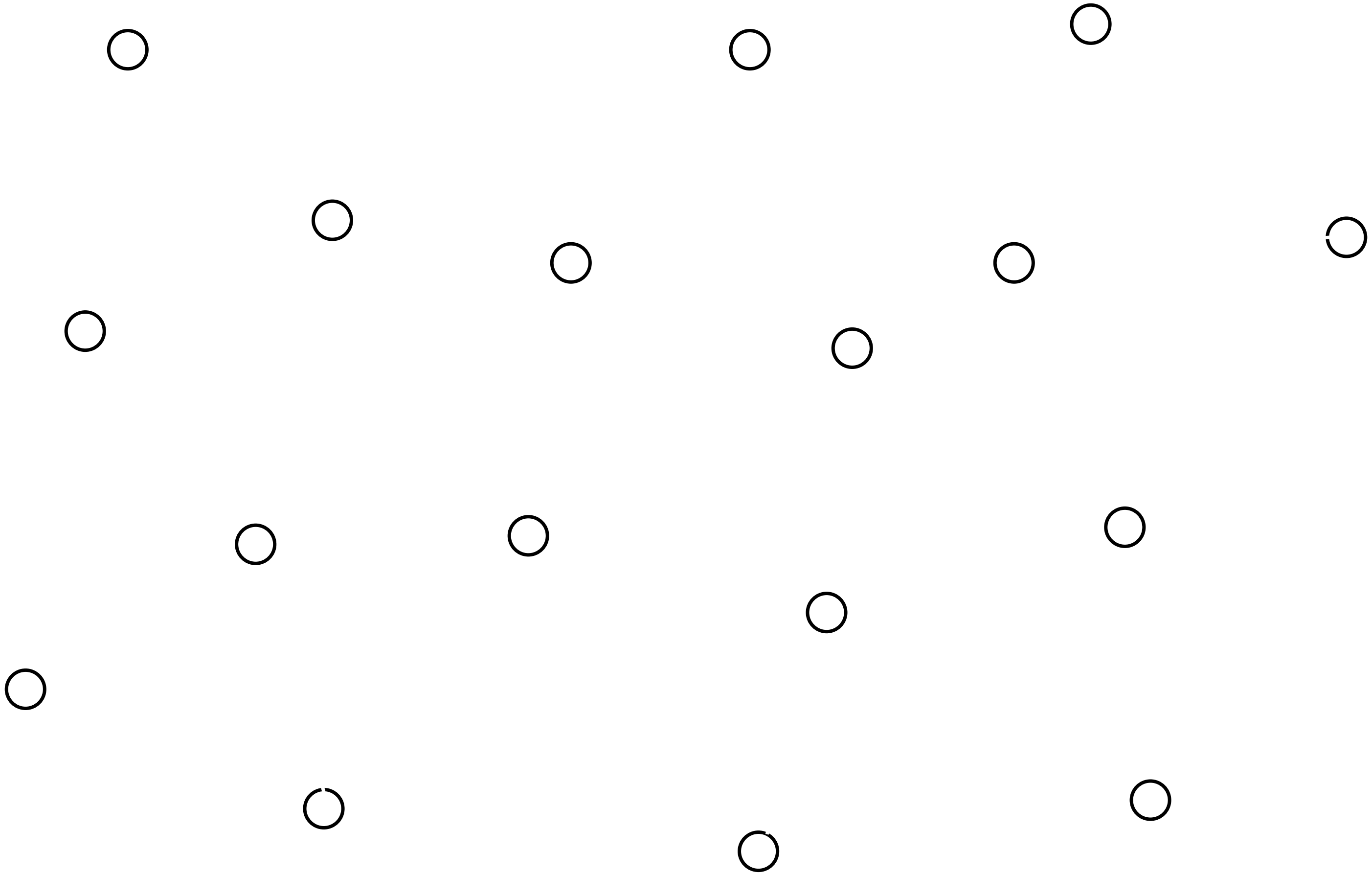
- Understand formation of large networks
- Internet, social networks, networks for exchanging goods are product of many selfish agents
- Emerge from distributed uncoordinated spontaneous actions
- How costly is lack of coordination?

Network creation game

- n agents build connected network
- Agent i lays out set of edges to other agents
- Edges may by used in both directions
- Hardware cost   QoS cost

Fabrikant, Luthra, Maneva, Papadimitriou, Shenker PODC'03

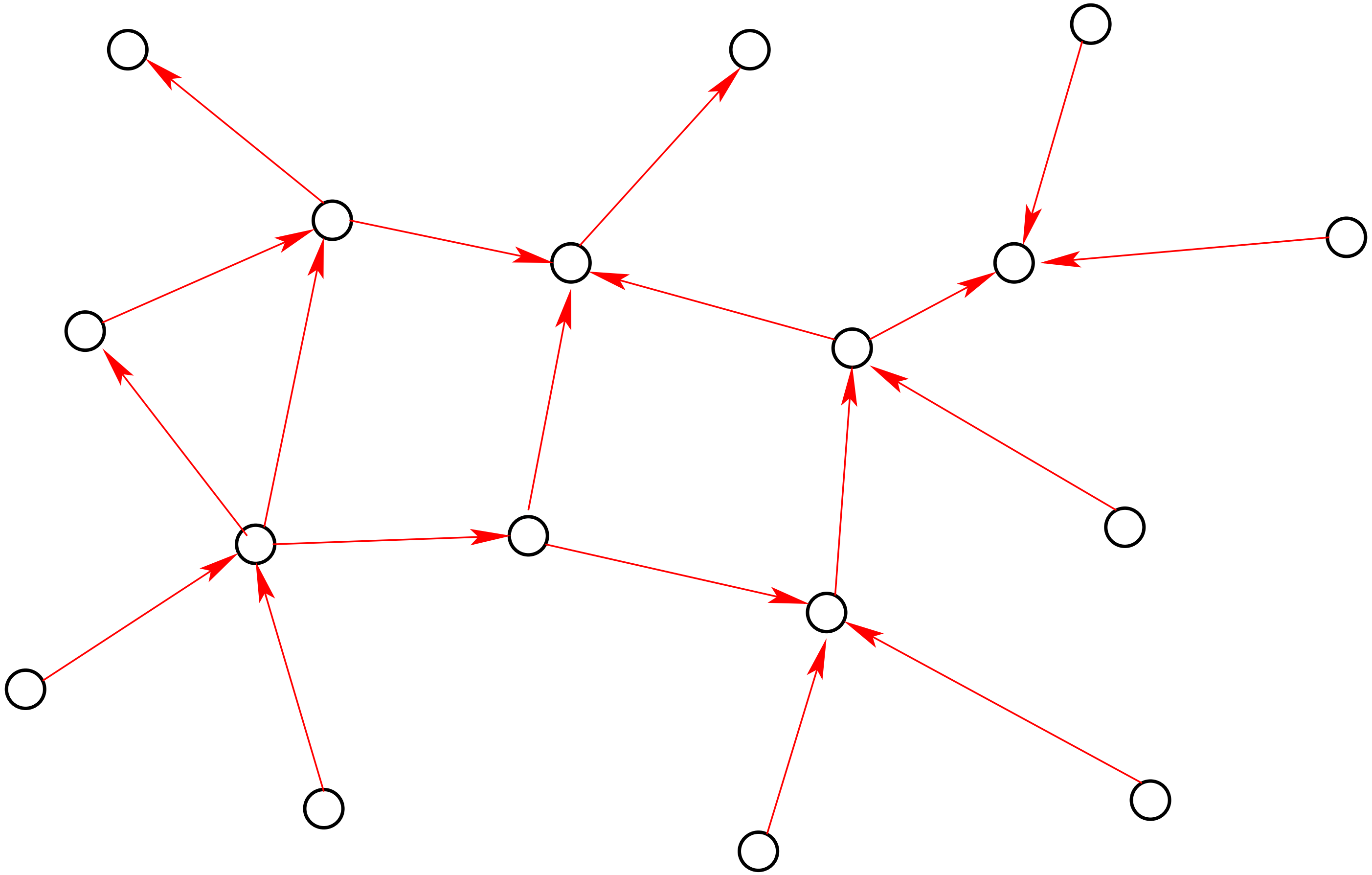
Network creation game



n agents have to build connected network.

Fabrikant, Luthra, Maneva, Papadimitriou, Shenker PODC'03

Network creation game



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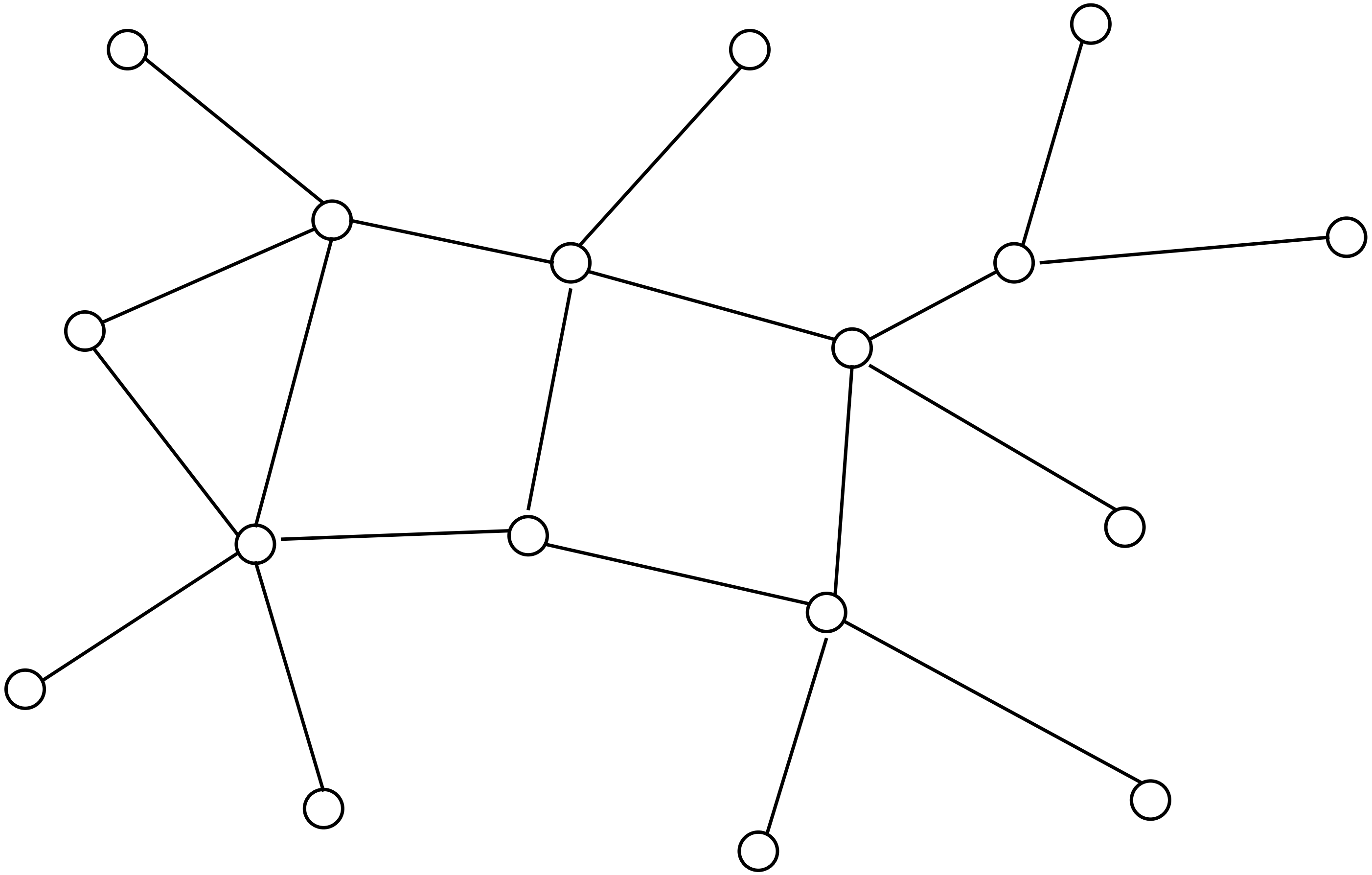
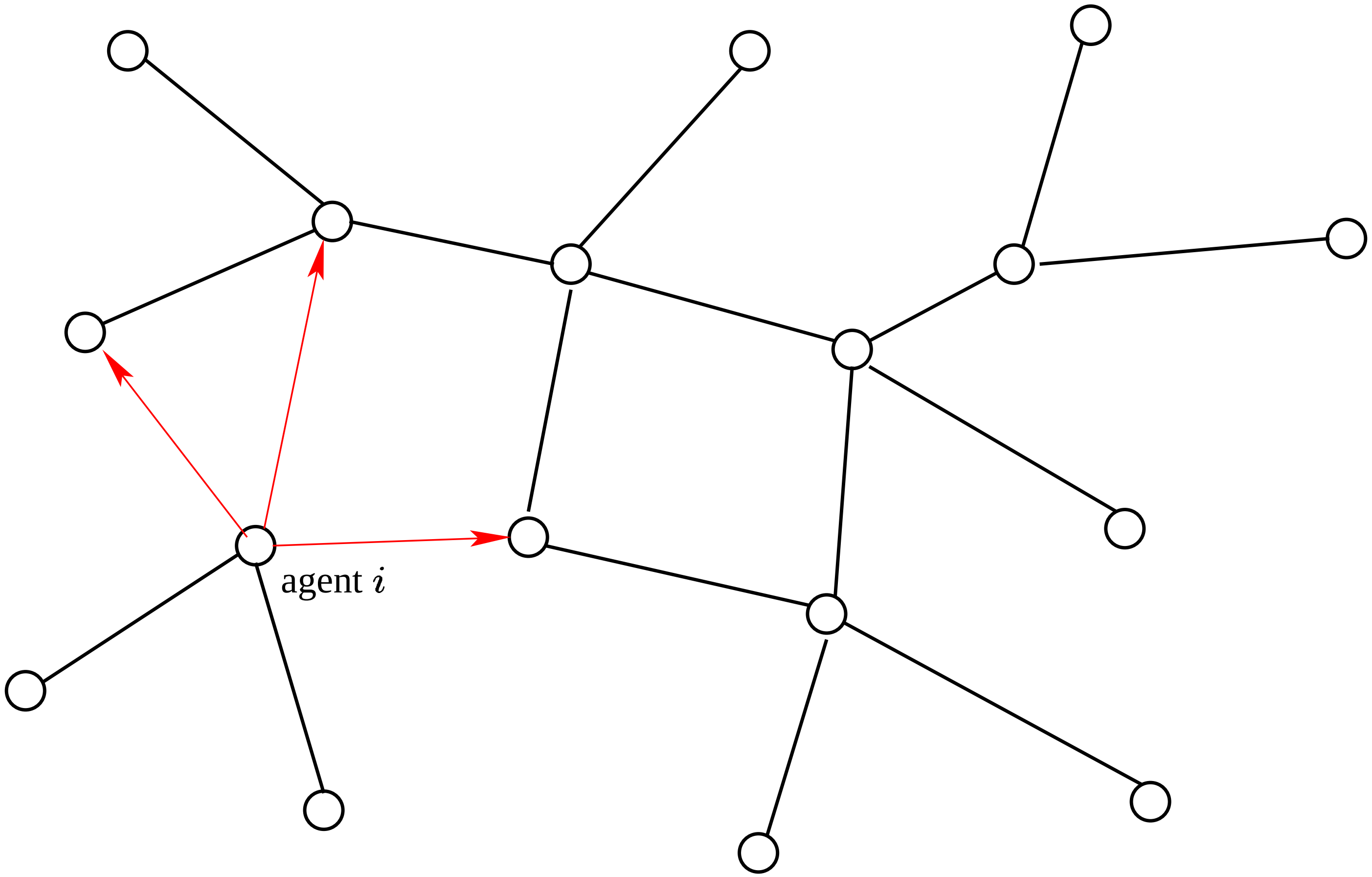
Network creation game



n agents have to build a connected network.

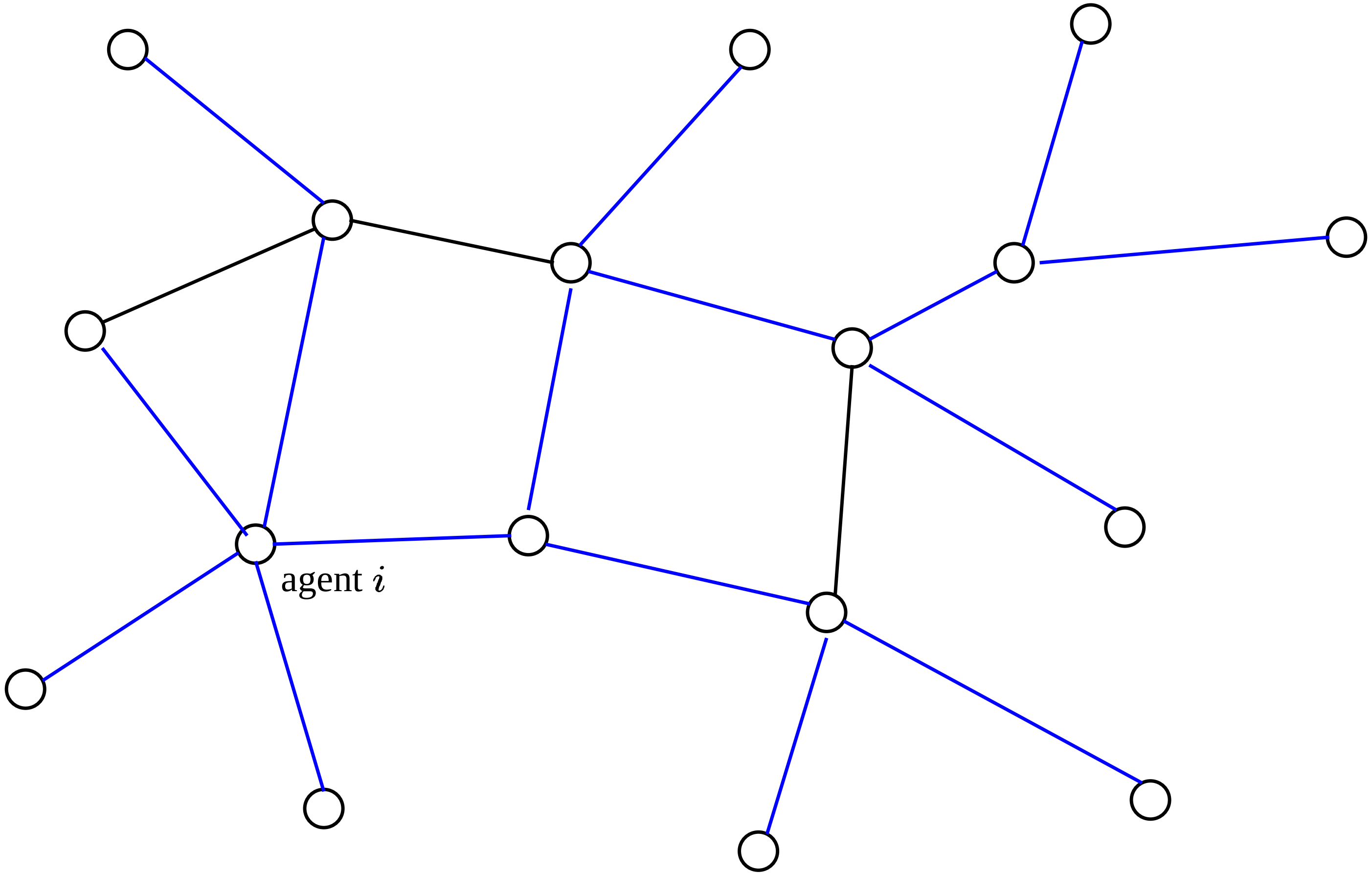
Fabrikant, Luthra, Maneva, Papadimitriou, Shenker PODC'03

Hardware cost



Cost of $\alpha > 0$ for each edge.

QoS cost



Shortest path distance to agent j, for all $j \neq i$.

Applications

- Information services networks

  Nodes represent facilities containing data storage; data has to be replicated to other nodes

- Social networks

  Set of nodes represents a community for disseminating information; edges represent phone calls.

- Postal and delivery services

  Nodes represent mail office branches; a link indicates that mail can reach endpoints directly

Related work

- Edge installation incurs cost but also yields a benefit;
  links may fail with certain probability.
  Bala, Goyal 2000; Haller, Sarangi 2000

- Agents receive payments but have costs for routing through traffic
  Johari, Mannor, Tsitsiklis 2005

- Edges are formed if both endpoints agree
  Corbo, Parkes 2005

Problem

Agents $V = \{1, \ldots, n\}$

Strategy of agent i $\quad S_i \subseteq V \setminus \{i\}$

Combination of strategies $\quad \vec{S} = (S_1, \ldots, S_n)$

$$\begin{aligned} G &= (V, E) \\ E &= \bigcup_{i \in V} \bigcup_{j \in S_i} \{i, j\} \end{aligned}$$

$$\begin{aligned} \text{Cost}(i, \vec{S}) &= \alpha\, |S_i| + \sum_{\substack{j \in V \\ j \neq i}} \text{Dist}(i, j) \\ \text{Cost}(\vec{S}) &= \sum_{i=1}^{n} \text{Cost}(i, \vec{S}) \end{aligned}$$

Nash equilibria

$\vec{S}$ forms Nash equilibrium if, for all i,

$$\text{Cost}(i, \vec{S}) \leq \text{Cost}(i, \vec{S}')$$

for all $\vec{S}'$ that differ from $\vec{S}$ only in i-th component

$\vec{S}$ is strong if inequality is strict; otherwise weak.

$\vec{S}$ is transient if there is a sequence of single-agent strategy changes leading to non-equilibrium state.

Price of anarchy

$$P = \max_{\vec{S}\ \text{Nash eq.}} \frac{\text{Cost}(\vec{S})}{\text{Cost(OPT)}}$$

Cost(OPT): social optimum

Koutsoupias, Papadimitriou ’99

Previous results

Fabrikant, Luthra, Maneva, Papadimitriou, Shenker PODC'03

- Computing optimal strategy for an agent is NP-hard
- $\alpha < 1$, $\alpha > n^2$: P is constant
- $1 \leq \alpha \leq n^2$: P is bounded by $O(\sqrt{\alpha})$
- Lower bound: $P \geq 3$
- Tree-conjecture: $\exists C\ \forall \alpha > C$ every non-transient Nash equilibrium is tree.
  If tree-conjecture holds, P is constant, for all α

Our results

Tree-conjecture is wrong: $\forall n$ $\exists$graph built by at least n agents that contains cycles and forms strong Nash equilibrium for $1 < \alpha \leq \sqrt{n/3}$

$P = O(1 + (\min\{\frac{\alpha^2}{n}, \frac{n^2}{\alpha}\})^{1/3})$ P constant for $\alpha \geq 12n \log n$

$\alpha \in O(\sqrt{n})$: P is constant

$\alpha \in \Omega(\sqrt{n})$, $\alpha \in O(n)$: P increasing, bounded by $O(n^{1/3})$

$\alpha \in \Omega(n)$: P decreasing, constant for $\alpha \geq 12n \log n$

Our results

Upper bounds can be extended to:

Cost sharing: agent can pay for a fraction of an edge

Weighted game: t_{ij} = traffic sent from agent i to j

$$\text{Cost}(i, \vec{S}) = \alpha|S_i| + \sum_{j \neq i} t_{ij}\text{Dist}(i, j)$$

Nash equilibrium representing a chordal graph is transient.

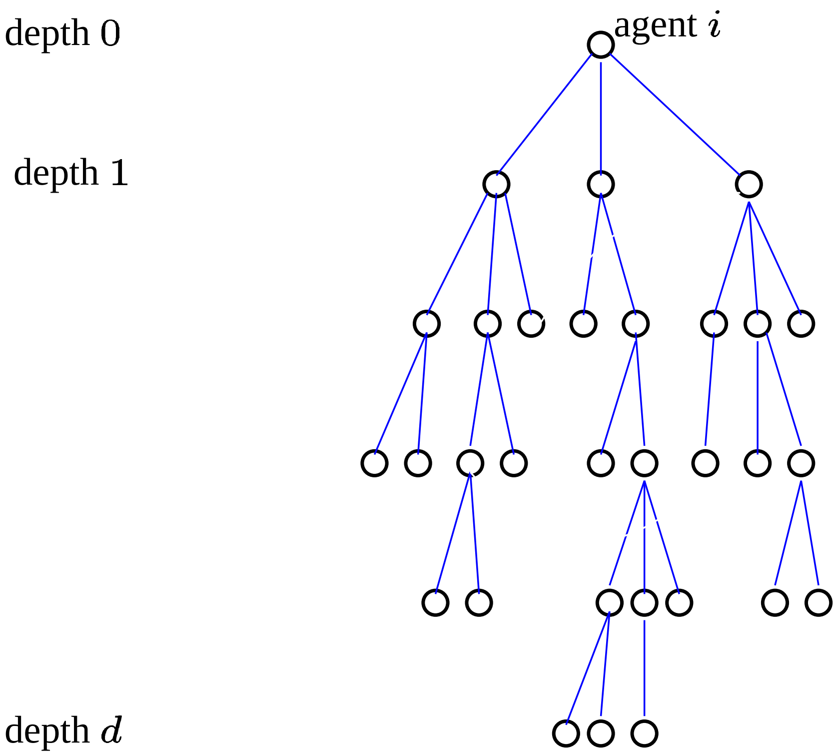
Such Nash equilibria exist for any n.

In any Nash equilibrium hardware cost is at most $2\text{Cost}(OPT)$.

Upper bound

Nash equilibrium $\vec{S}$ $\quad G = (V, E)$

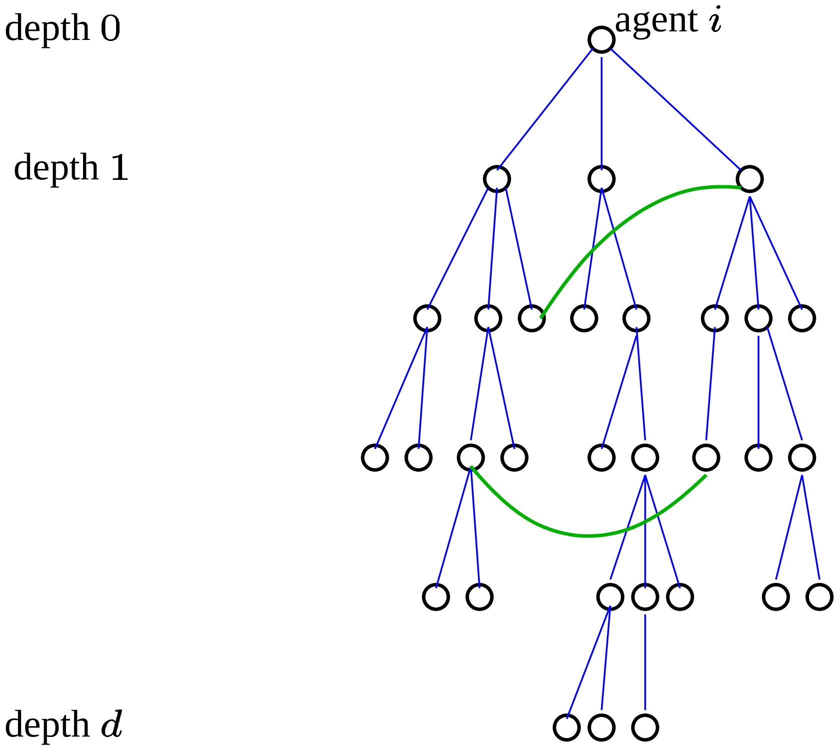
Shortest path tree rooted at agent i



Upper bound

Nash equilibrium $\vec{S}$ $G = (V, E)$

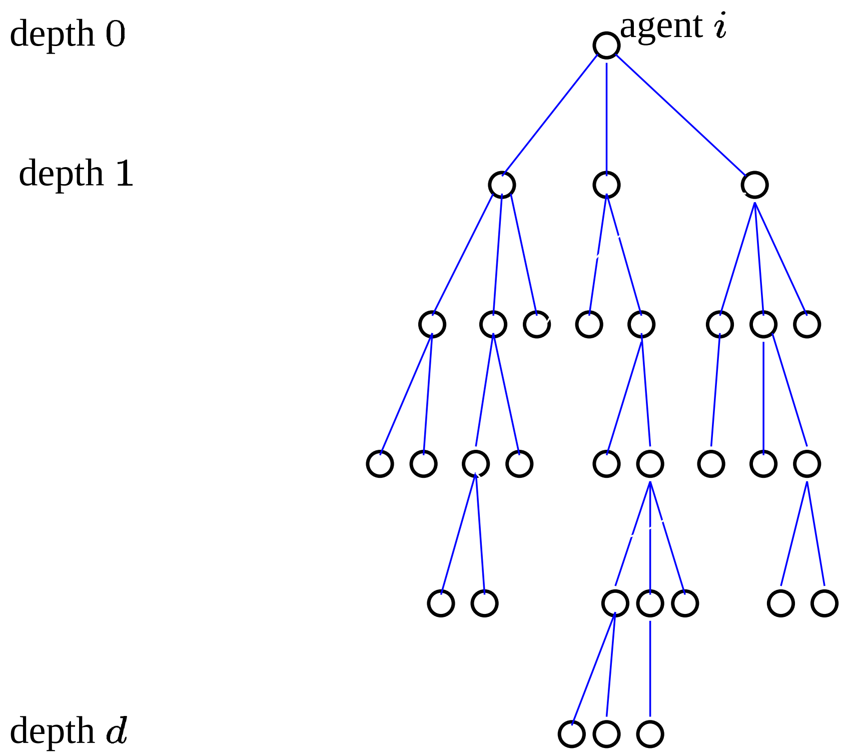
Shortest path tree rooted at agent i



Cost agent i

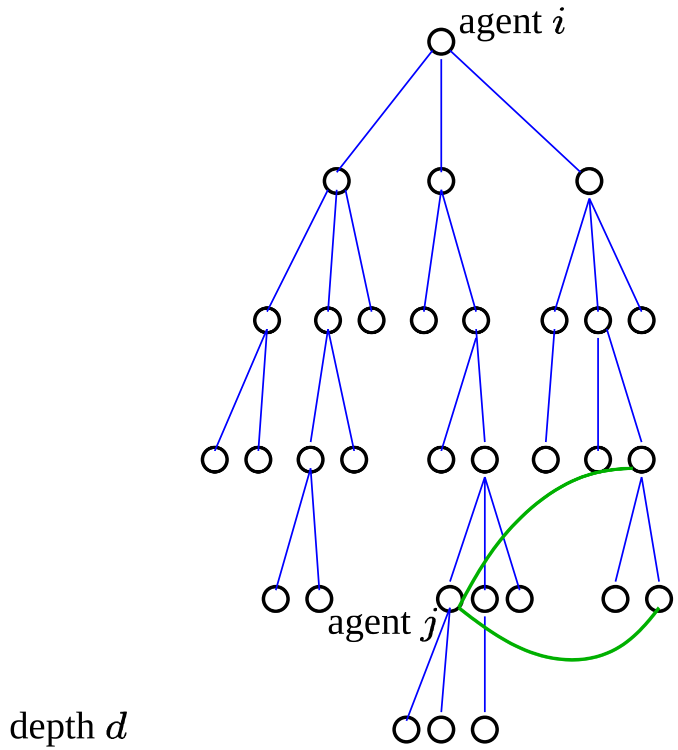
Nash equilibrium $\vec{S}$ $\quad G = (V, E)$

Shortest path tree rooted at agent i

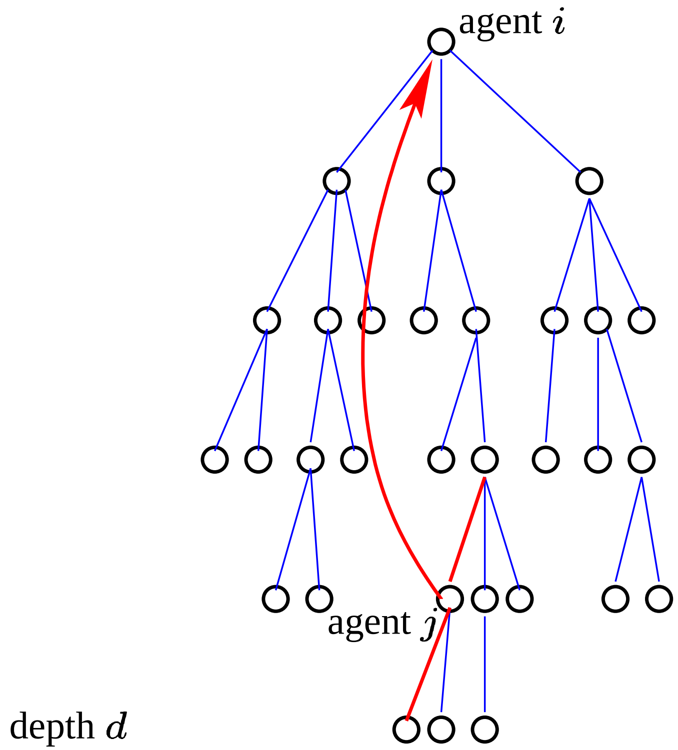


$\text{Cost}(i, \vec{S}) \leq \alpha T_i + d(n-1)$ $\qquad$ $T_i = \#$tree edges built by agent i

Cost of agent j

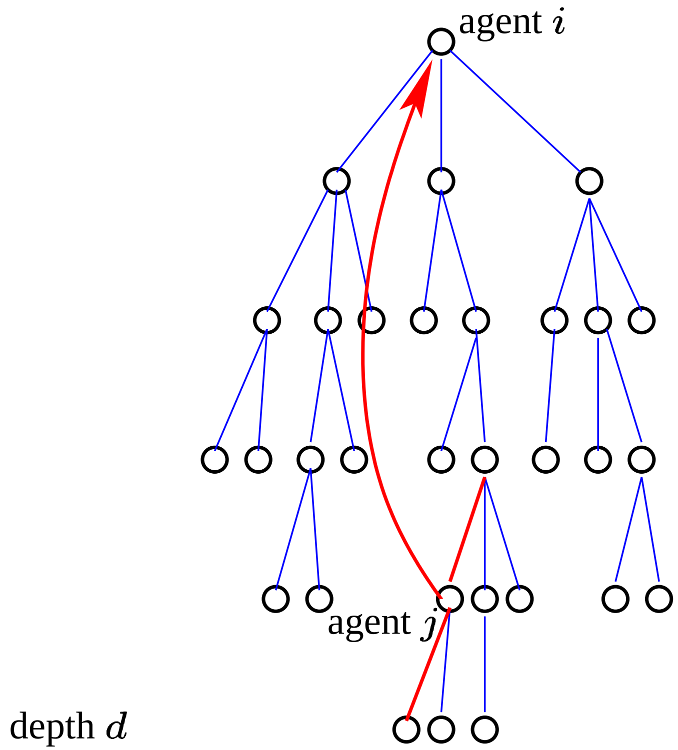


Cost of agent j



$\alpha T_j + \alpha$ + $(d+1)(n-1)$

Cost of agent j



$$\text{Cost}(j, \vec{S}) \leq \alpha T_j + \alpha + (d+1)(n-1)$$

Cost Nash

$$\text{Cost}(i, \vec{S}) \leq \alpha T_i + d(n-1)$$

$$\text{Cost}(j, \vec{S}) \leq \alpha T_j + \alpha + (d+1)(n-1) \qquad \forall j \neq i$$

$$\text{Cost}(\vec{S}) \leq \alpha(n-1) + \alpha(n-1) + (d+1)n(n-1)$$

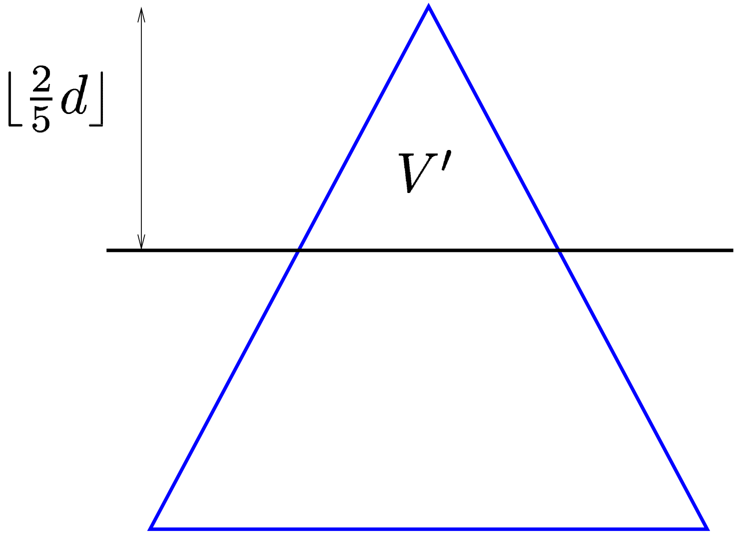
Price of anarchy

$$\text{Cost}(\vec{S}) \leq 2\alpha(n-1) + (d+1)n(n-1)$$

$$\text{Cost(OPT)} \geq \alpha(n-1) + n(n-1)$$

Analysis d

$V' = \{j \mid j$ has depth at most $\lfloor \frac{2}{5} d \rfloor$ in shortest path tree$\}$



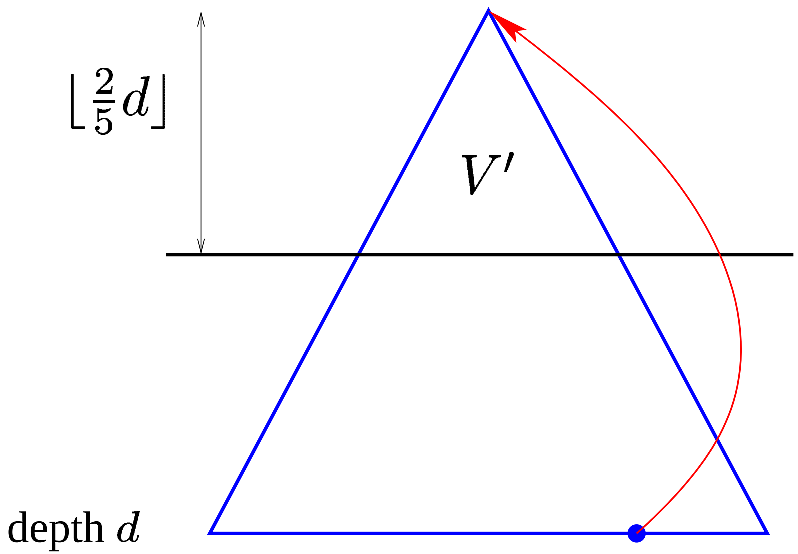
Case 1: $|V'| \geq \frac{2}{3} n^c$

Case 2: $|V'| < \frac{2}{3} n^c$

c s.t. $\alpha = n^{3c-1}$

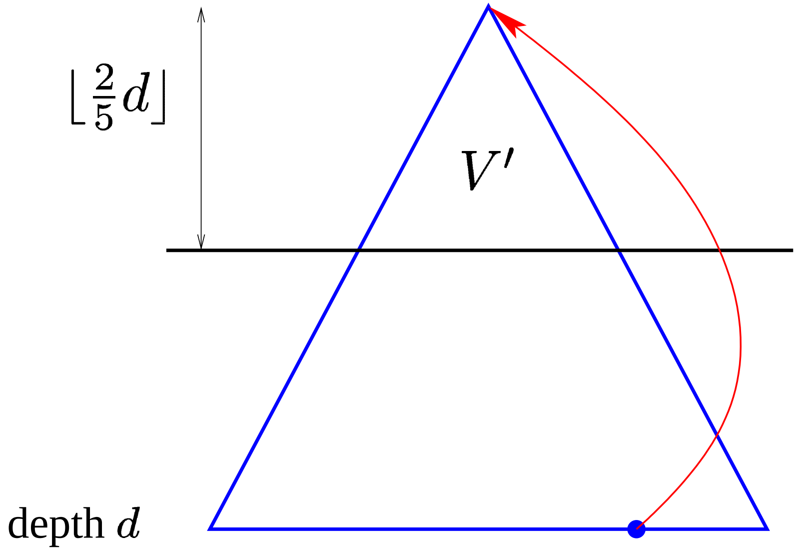
$1/3 \leq c \leq 1$

Case 1: $|V'| \geq \frac{2}{3}n^c$



$$|V'| \left(\left\lceil \frac{3}{5}d \right\rceil - \left\lfloor \frac{2}{5}d \right\rfloor - 1 \right)$$

Case 1: $|V'| \geq \frac{2}{3}n^c$

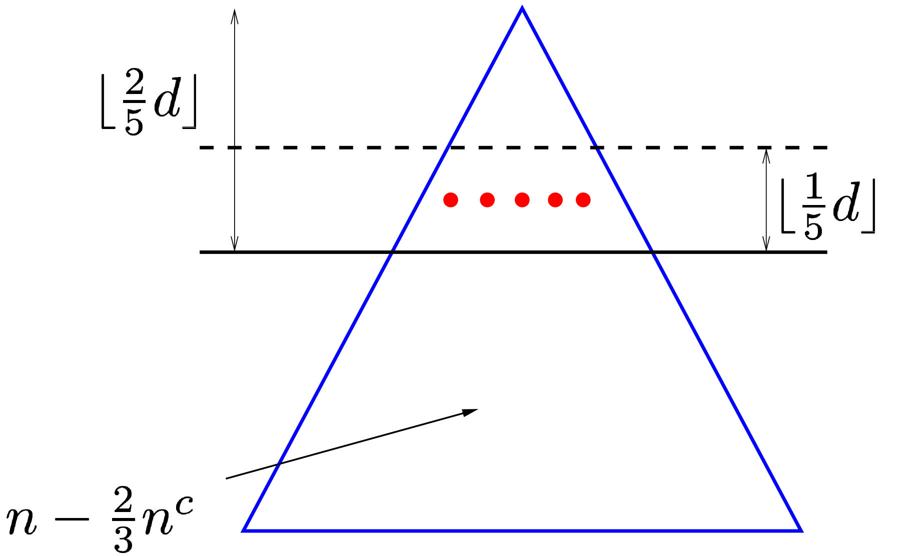


$$\alpha \geq |V'|\left(\left\lceil \frac{3}{5}d \right\rceil - \left\lfloor \frac{2}{5}d \right\rfloor - 1\right) \geq |V'|\left(\frac{1}{5}d - 1\right)$$

$$d \leq \frac{15\alpha}{n^c}$$

Case 2: $|V'| < \frac{2}{3}n^c$

$\exists d_0$ having at most $\frac{2}{3}n^c / \lfloor \frac{1}{5}d \rfloor$ vertices



Case 2: $|V'| < \frac{2}{3}n^c$

$$\alpha\left(\frac{2}{3}n^c / \left\lfloor \frac{1}{5}d \right\rfloor\right) \geq \left\lfloor \frac{1}{5}d \right\rfloor \left(n - \frac{2}{3}n^c\right)$$

$$d \leq \frac{15\alpha}{n^c}$$

Price of anarchy

$$\text{Cost}(\vec{S}) \leq 2\alpha(n-1) + (15\alpha/n^c + 1)n(n-1)$$

$$\text{Cost(OPT)} \geq \alpha(n-1) + n(n-1)$$

$$\alpha = n^{3c-1} \Longrightarrow n^c = (\alpha n)^{1/3}$$

$$P = O(1 + (\min\{\tfrac{\alpha^2}{n}, \tfrac{n^2}{\alpha}\})^{1/3})$$

Disproving tree conjecture

$\forall n$ $\exists$graph built by at least n agents that contains cycles and forms strong Nash equilibrium for $1 < \alpha \leq \sqrt{n/3}$

Construct geodetic graphs of diameter 2

Geodetic: shortest path between any pair of points is unique

Disproving tree conjecture

Affine plane $(A, \mathcal{L})$

A set of points $\mathcal{L}$ set of lines

- Any two points are contained in exactly one line.
- Each line contains at least two points.
- For any point x and any line L not containing x, there is a unique line L' that contains x and is disjoint from L.
- There exists a triangle, i.e. there are three points not contained in a line.

Two lines are parallel if disjoint or equal

Parallelism defines equivalence relation on $\mathcal{L}$

Affine plane

Affine plane $AG(2, q)$, q prime power

- $F = GF(q)$ finite field of order q.
- $A = F^2$ and $\mathcal{L} = \{a + bF \mid a, b \in A, b \neq 0\}$

Properties

- Each line contains q points
- q^2 points and $q(q + 1)$ lines
- $q + 1$ equivalence classes with q lines

Graph

Affine plane $AG(2, q)$

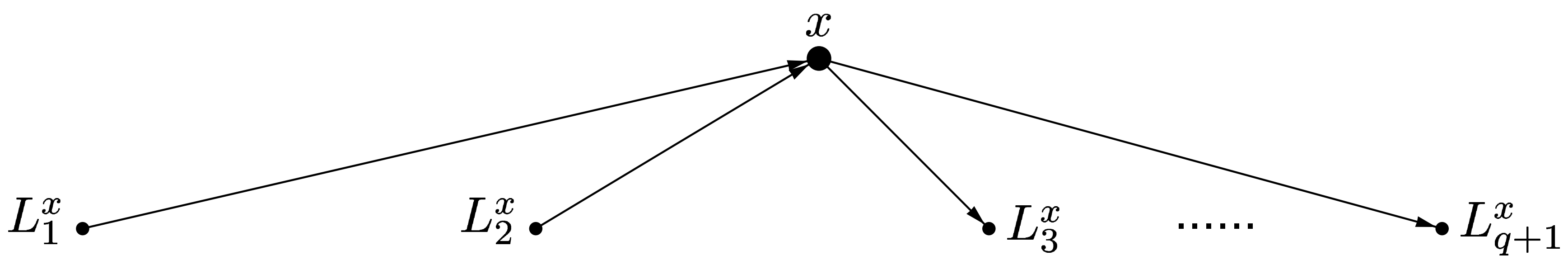
$G = (V, E)$ mit $V = A \cup \mathcal{L}$

- L and L' connected if $L \parallel L'$
- x and L connected if $x \in L$

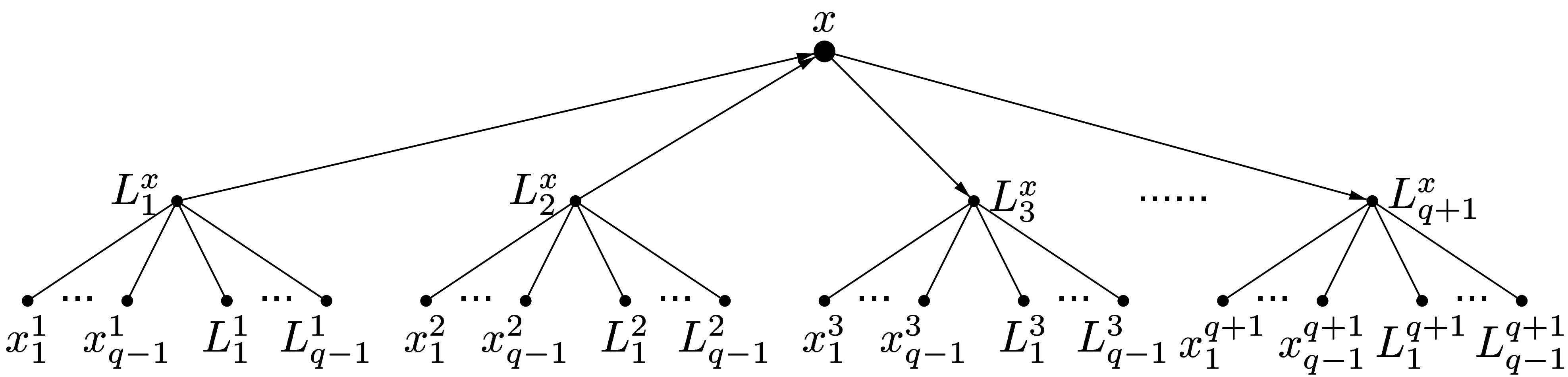
Orientations

- Edges of $[L]$ form K_q. $|indeg(L') - outdeg(L')| \leq 1$
- $x \in L_i^q$ has incoming edges from lines in i-th and $(i-1)$-st eq. classes that contain x.

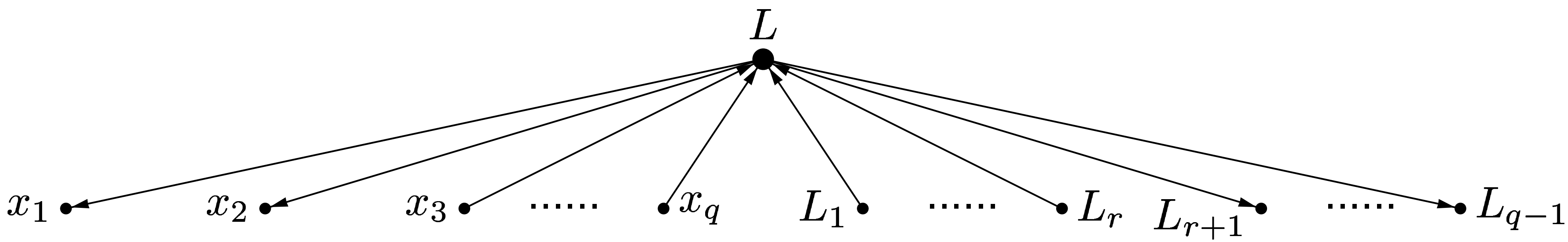
Player of a point



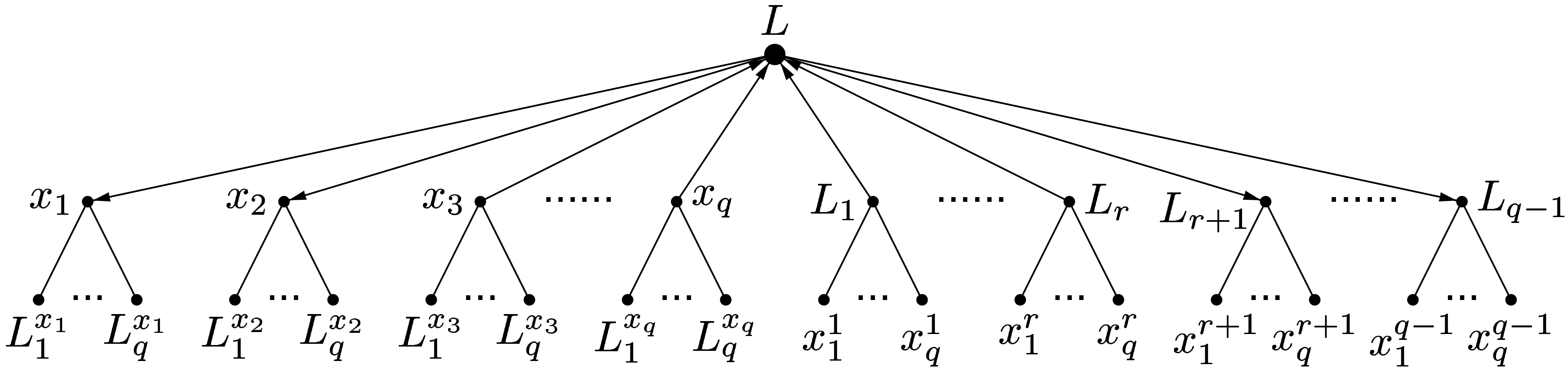
Player of a point



Player of a line

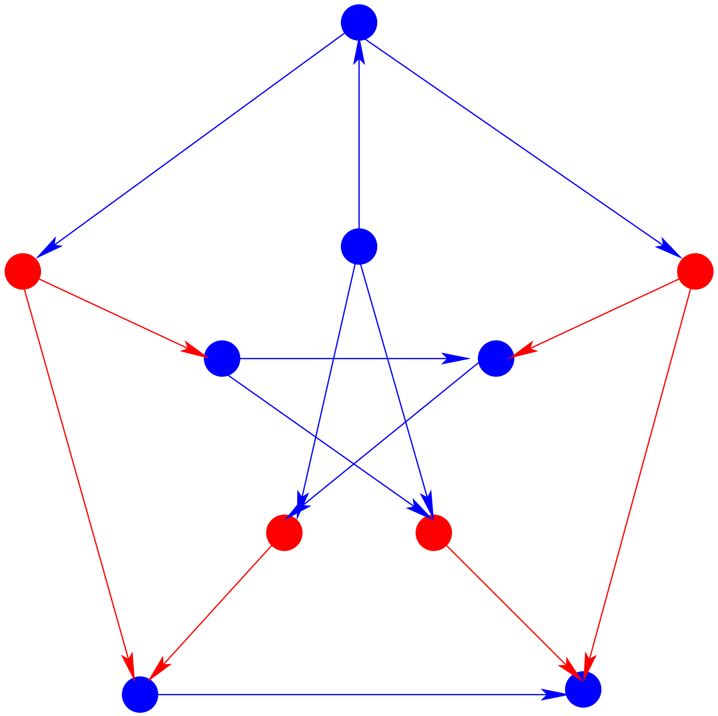


Player of a line



Petersen graph

$q = 2$



Open problems

Determine exact P, for any α

Determine upper bound on diameter

Study other network creation games