Constructing routing backbones in heterogeneous wireless ad-hoc networks

(Constant-factor approximation for minimum-weight (connected) dominating sets in unit disk graphs)

Thomas Erlebach

Joint work with: Christoph Ambühl (Liverpool), Matúš Mihal’ák (Leicester), Marc Nunkesser (ETH Zurich)
Where is Leicester?
Where is Leicester?
Disk graphs

...are the intersection graphs of disks in the plane:
Subclasses of disk graphs

**Coin graphs**: touching graphs of disks whose interiors are disjoint

Coin graphs are exactly the planar graphs! [Koebe, 1936]

**Unit disk graphs**: all disks have radius 1
UDG application: Wireless networks
**Minimum Dominating Set (MDS)**

**Input:** a set \( \mathcal{D} \) of unit disks in the plane

**Feasible solution:** subset \( A \subseteq \mathcal{D} \) that dominates all disks

**Goal:** minimize \(|A|\)
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In the **weighted** case (MWDS), each disk is associated with a positive weight.
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For **Minimum (Weight) Connected Dominating Set** (MCDS/MWCDS), the dominating set must induce a connected subgraph.
Motivation

Flooding is an important primitive in wireless (ad-hoc) networks.

To make flooding more efficient, compute a dominating set (routing backbone) and let only the nodes in that set forward the message, e.g. [Alzoubi et al., 2002]

Use a connected dominating set to ensure that all backbone nodes receive a message originating at any node.

Ad-hoc networks can be heterogeneous, so not all nodes are equally suitable for participating in the routing backbone.

Interest in weighted (connected) dominating set problems in unit disk graphs [Wang and Li, 2005]
Known results for MDS

- In arbitrary graphs, ratio $\Theta(\log n)$ is best possible (unless $P = NP$) for MDS, MWDS, MCDS and MWCDS. [Feige ’96; Arora and Sudan ’97; Guha and Khuller ’99]

- For MDS in unit disk graphs, a PTAS can be obtained using the shifting strategy [Hunt III et al., 1994]:
  - Any maximal independent set is a dominating set.
  - Therefore, the smallest dominating set in a constant-size square can be found in polynomial time by enumeration.

- PTAS also for MCDS in unit disk graphs [Cheng et al., 2003]

- Question:
  - MWDS and MWCDS in unit disk graphs?
Known PTASs

- Planar graphs:

- Unit disk graphs:

- Disk graphs:
Known PTASs

- **Planar graphs**: Shifting Strategy (Baker)
  - Max Weight Independent Set: PTAS ✔️ [B83]
  - Min Weight Vertex Cover: PTAS ✔️ [B83]
  - Min Weight Dominating Set: PTAS ✔️ [B83]

- **Unit disk graphs**: [Not specified]

- **Disk graphs**: [Not specified]
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- **Unit disk graphs**: Shifting (Hochbaum & Maass)
  - Max Weight Ind. Set: PTAS ✔ [HM85,HMR+98]
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  - Min Weight Dominating Set: ???

- **Disk graphs:**
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- **Disk graphs**: Shifting + Dynamic Programming
  - Max Weight Ind. Set: PTAS ✔ [EJS01]
  - Min Weight Vertex Cover: PTAS ✔ [EJS01]
  - Min Dominating Set: ???
  - Min Weight Dominating Set: ???
Can we use shifting for MWDS in UDG?

MWDS can be arbitrarily large for unit disks in an area of constant size:

Brute-force enumeration no longer works.
Main result

**Theorem.** There is a constant-factor approximation algorithm for MWDS in unit disk graphs.

**Ideas:**

- Partition the plane into unit squares and solve the problem for each square separately.
- In each square, reduce the problem to the problem of covering points with weighted disks.
- Use enumeration techniques (guess properties of $\text{OPT}$) and dynamic programming to solve the latter problem.

The constant factor is 72.
The subproblem for each square

Find a dominating set for the square:
- Let $D_S$ denote the set of disks with center in a $1 \times 1$ square $S$.
- Let $N(D_S)$ denote the disks in $D_S$ and their neighbors.
- Task: Find a minimum weight set of disks in $N(D_S)$ that dominates all disks in $D_S$. 
The subproblem for each square

- Find a dominating set for the square:
  - Let $\mathcal{D}_S$ denote the set of disks with center in a $1 \times 1$ square $S$.
  - Let $N(\mathcal{D}_S)$ denote the disks in $\mathcal{D}_S$ and their neighbors.
  - **Task:** Find a minimum weight set of disks in $N(\mathcal{D}_S)$ that dominates all disks in $\mathcal{D}_S$.

Reduces (by guessing the max weight of a disk in $\text{OPT}_S$) to covering points in a square with weighted disks:

- Let $P$ be a set of points in a $\frac{1}{2} \times \frac{1}{2}$ square $S$.
- Let $\mathcal{D}$ be a set of weighted unit disks covering $P$.
- **Task:** Find a minimum weight set of disks in $\mathcal{D}$ that covers all points in $P$. 
Covering points by weighted disks
Covering points by weighted disks

Remark. $O(1)$-approximation algorithms are known for unweighted disk cover [Brönninmann and Goodrich, 1995].
Given a set of points in a strip, and a set of weighted unit disks with centers outside the strip, compute a minimum weight set of disks covering the points.
Dynamic programming

Vertical sweepline, table entry for every pair of disks that could be on the lower and upper envelope:
Main cases: One hole or many holes

One-hole case:

Enlarged:

Many-holes case:

Enlarged:
**Sketch of the one-hole case**

**Step 1:** Guess the four “corner points” of the optimal solution (each of them is defined by two disks).
Sketch of the one-hole case

Step 2: Two regions that can only be covered with disks whose centers are to the left or right of the square.
Sketch of the one-hole case

Step 3: Remaining area can only be covered with disks whose centers are above or below the square.
Summary: MWDS in unit disk graphs

- Partition the plane into unit squares and solve the problem for each square separately. (We lose a factor of 36 compared to OPT.)
- For each square, reduce the weighted dominating set problem to a weighted disk cover problem.
- Distinguish one-hole case and many-holes case.
- In each case, we have a 2-approximation or optimal algorithm for covering points in the square with weighted unit disks.
- This implies the constant-factor approximation algorithm for MWDS in unit disk graphs. The ratio is \(36 \cdot 2 = 72\).
Weighted Connected Dominating Sets

**Theorem.** There is a constant-factor approximation algorithm for MWCDS in unit disk graphs.

**Algorithm Sketch:**

1. First, compute an $O(1)$-approximate MWDS $D$.
2. Build auxiliary graph $H$ with a vertex for each component of $D$, and weighted edges corresponding to paths with at most two internal vertices.
3. Compute a minimum spanning tree of $H$ and add the disks corresponding to its edges to $D$.

We can show: The total weight of the disks added to $D$ is at most $17 \cdot \text{OPT}$, where OPT is the weight of a minimum weight connected dominating set.
Conclusion

- We have presented the first constant-factor approximation algorithms for MWDS and MWCDS in unit disk graphs.

- The approximation ratios are 72 and 89, respectively.

- The MWDS algorithm guesses many properties of the optimal solution and thus has a rather large running-time (exponent $> 10$)

Open Problems

- Improve approximation ratio and/or running-time.

- What approximation ratio can be achieved for weighted or unweighted dominating sets in arbitrary disk graphs?
Thank you!