Constructing routing backbones in heterogeneous wireless ad-hoc networks

(Constant-factor approximation for minimum-weight (connected) dominating sets in unit disk graphs)

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Where is Leicester?

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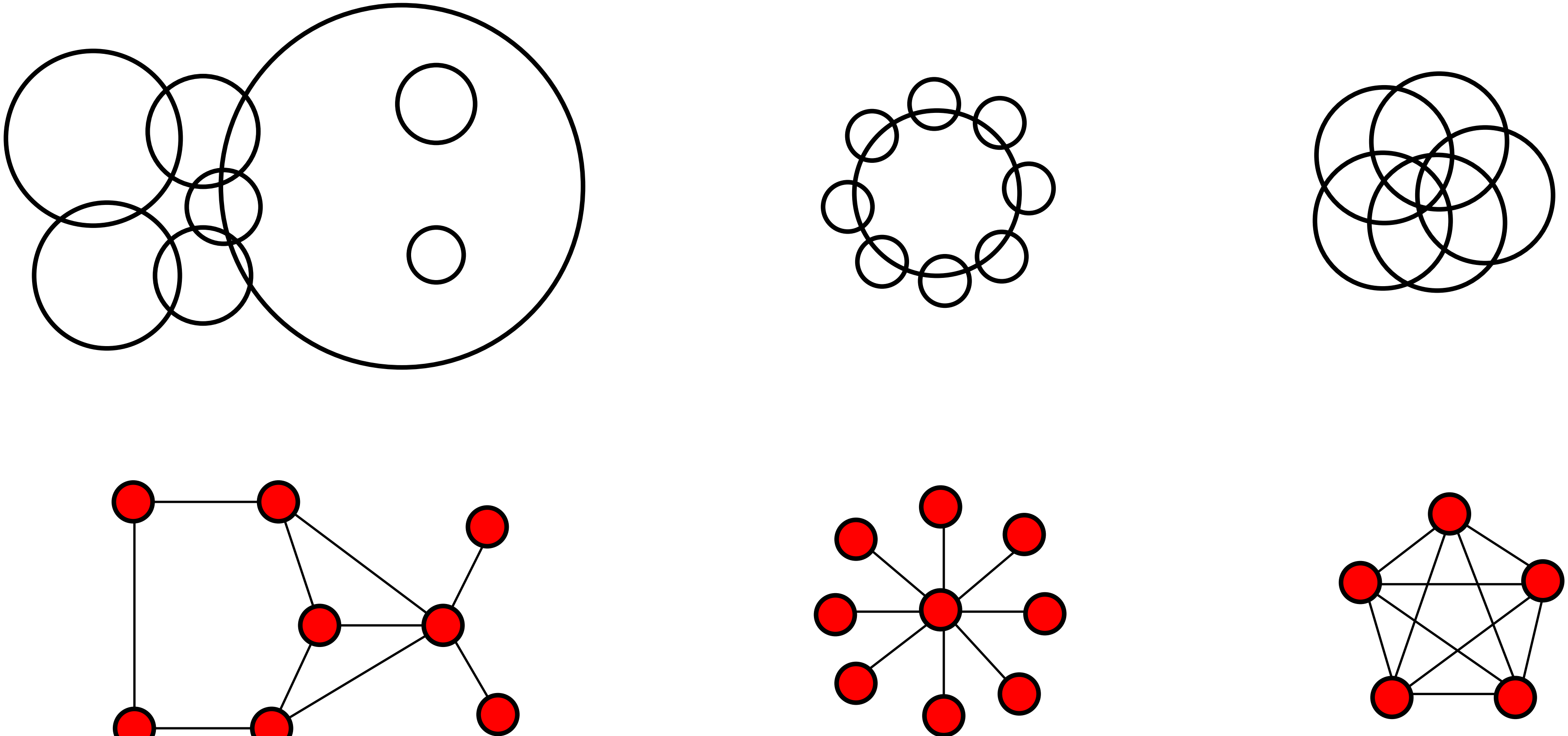
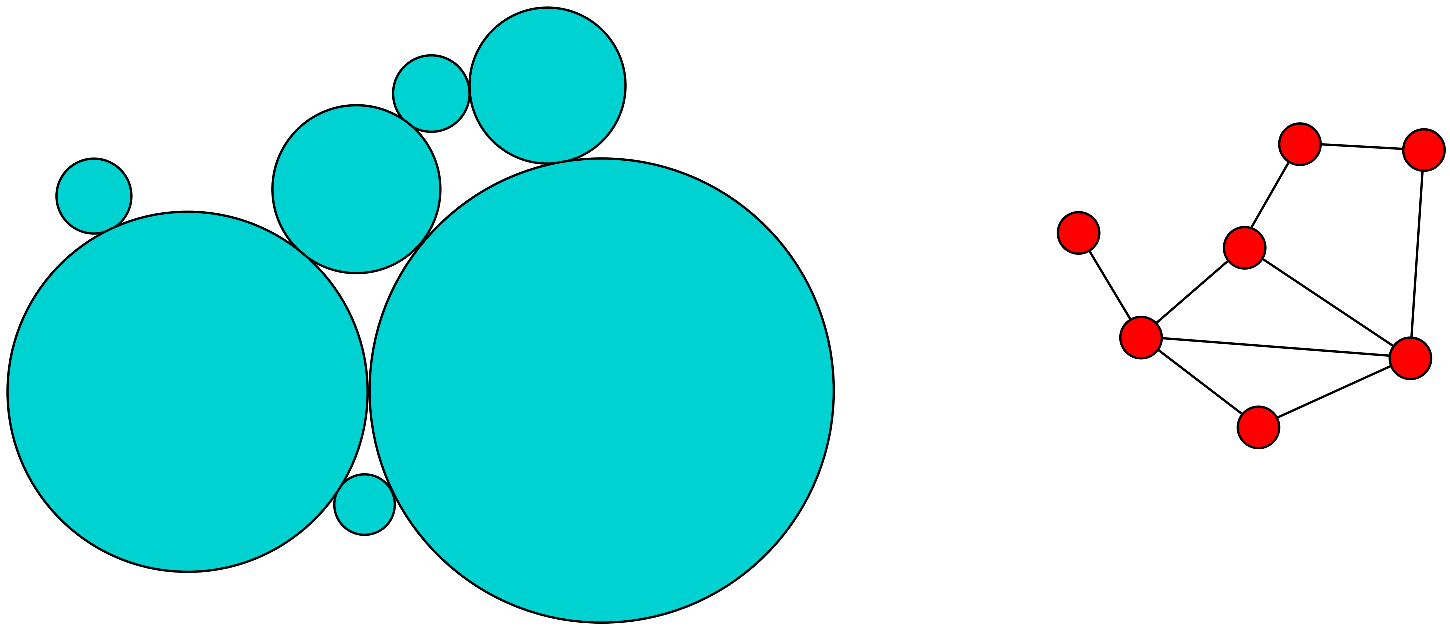
Disk graphs

. . . are the intersection graphs of disks in the plane:



Subclasses of disk graphs

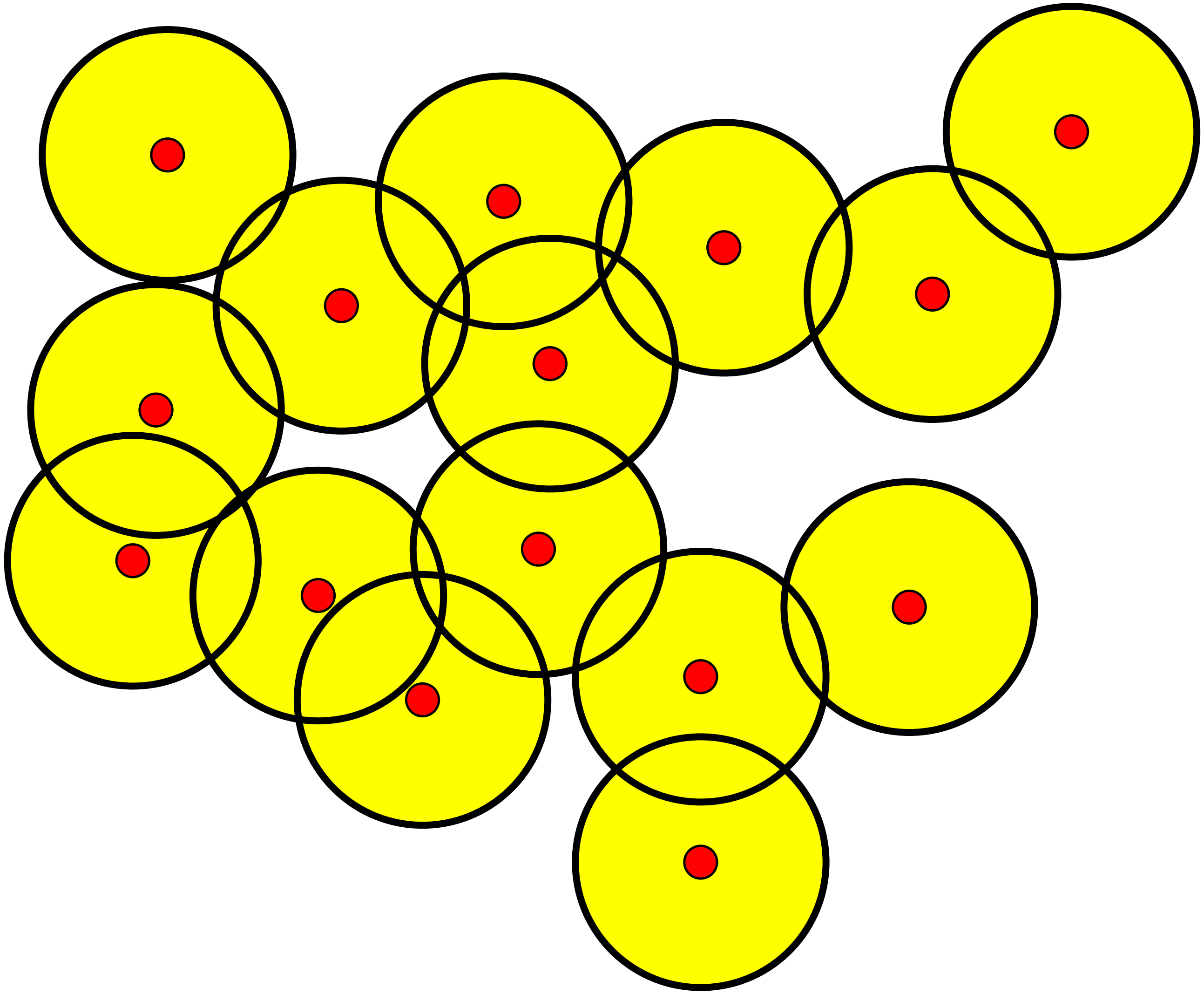
- **Coin graphs**: touching graphs of disks whose interiors are disjoint



Coin graphs are exactly the planar graphs! [Koebe, 1936]

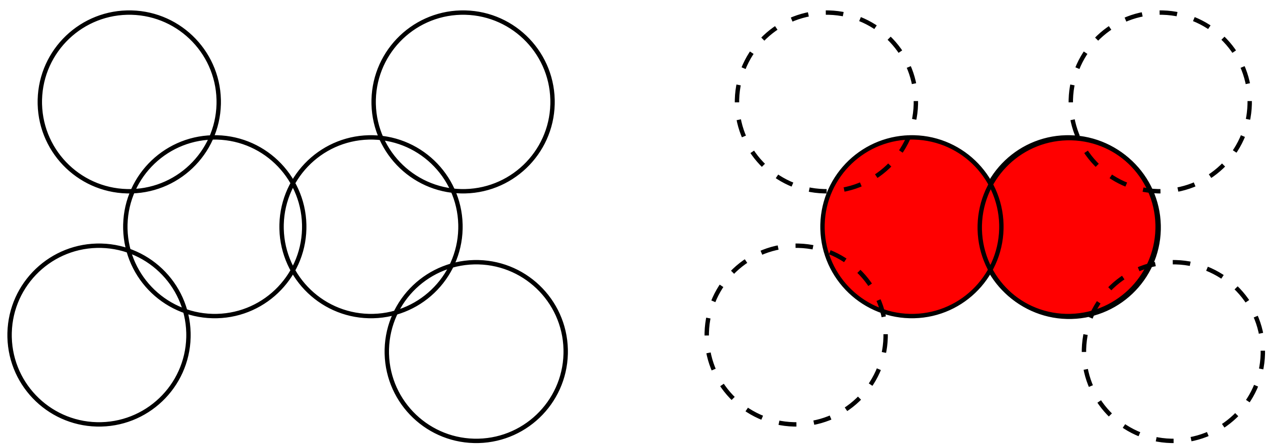
- **Unit disk graphs**: all disks have radius 1

UDG application: Wireless networks



Minimum Dominating Set (MDS)

Input: a set $\mathcal{D}$ of unit disks in the plane
Feasible solution: subset $A \subseteq \mathcal{D}$ that dominates all disks
Goal: minimize $|A|$



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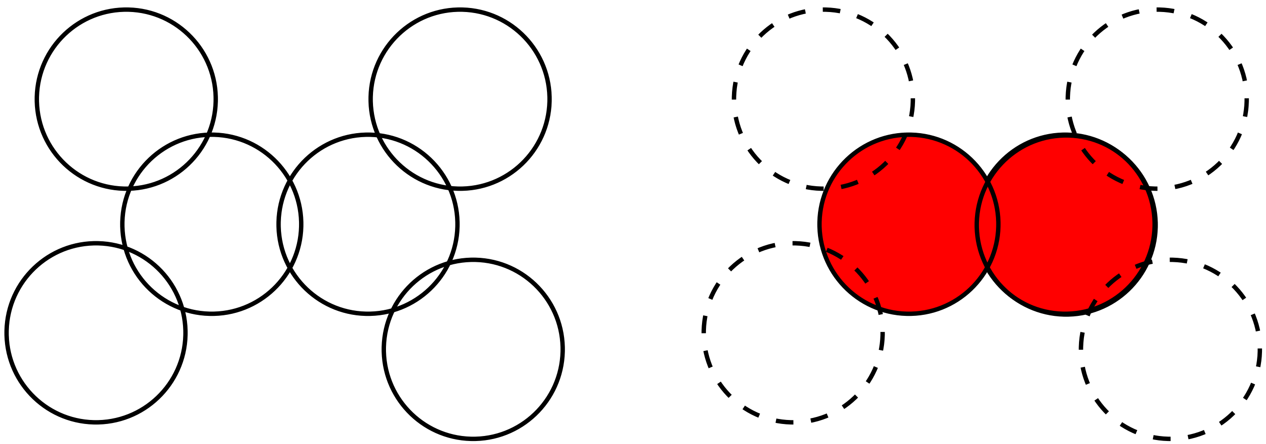

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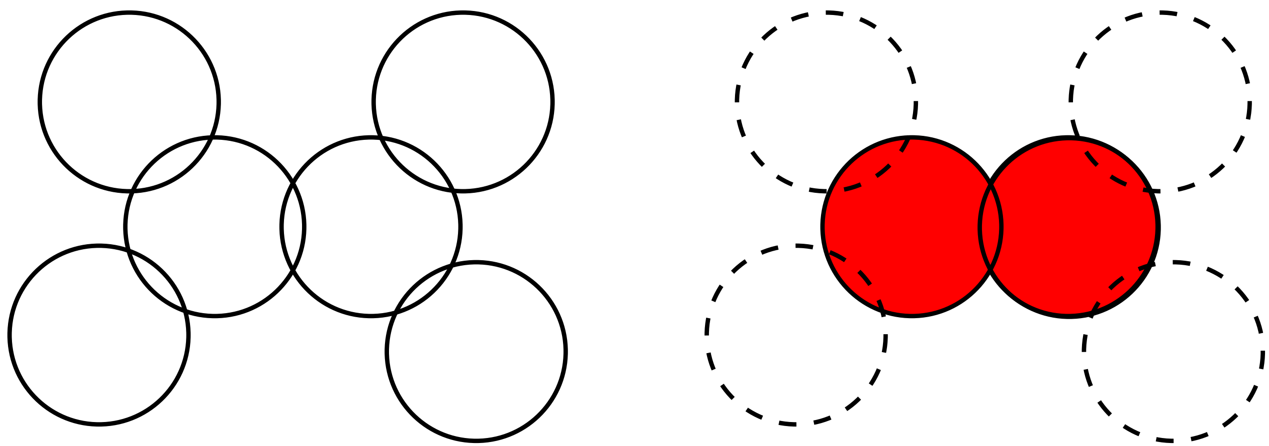

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For Minimum (Weight) Connected Dominating Set (MCDS/MWCDS), the dominating set must induce a connected subgraph.

Motivation

- Flooding is an important primitive in wireless (ad-hoc) networks.
- To make flooding more efficient, compute a dominating set (routing backbone) and let only the nodes in that set forward the message, e.g. [Alzoubi et al., 2002]
- Use a connected dominating set to ensure that all backbone nodes receive a message originating at any node.
- Ad-hoc networks can be heterogeneous, so not all nodes are equally suitable for participating in the routing backbone
  ➠ interest in weighted (connected) dominating set problems in unit disk graphs [Wang and Li, 2005]

Known results for MDS

- In **arbitrary graphs**, ratio $\Theta(\log n)$ is best possible (unless $P = NP$) for MDS, MWDS, MCDS and MWCDS. [Feige '96; Arora and Sudan '97; Guha and Khuller '99]
- For **MDS in unit disk graphs**, a PTAS can be obtained using the shifting strategy [Hunt III et al., 1994]:
  - Any maximal independent set is a dominating set.
  - Therefore, the smallest dominating set in a constant-size square can be found in polynomial time by enumeration.
- PTAS also for **MCDS in unit disk graphs** [Cheng et al., 2003]
- **Question:**
  - MWDS and MWCDS in unit disk graphs?

Known PTASs

- **Planar graphs:**

- **Unit disk graphs:**

- **Disk graphs:**

Known PTASs

- **Planar graphs:** Shifting Strategy (Baker)
  - Max Weight Independent Set: PTAS ✔ [B83]
  - Min Weight Vertex Cover: PTAS ✔ [B83]
  - Min Weight Dominating Set: PTAS ✔ [B83]
- **Unit disk graphs:**
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- **Unit disk graphs:** Shifting (Hochbaum & Maass)
  - Max Weight Ind. Set: PTAS ✔ [HM85,HMR+98]
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  - Min Weight Dominating Set: **???**
- **Disk graphs:**

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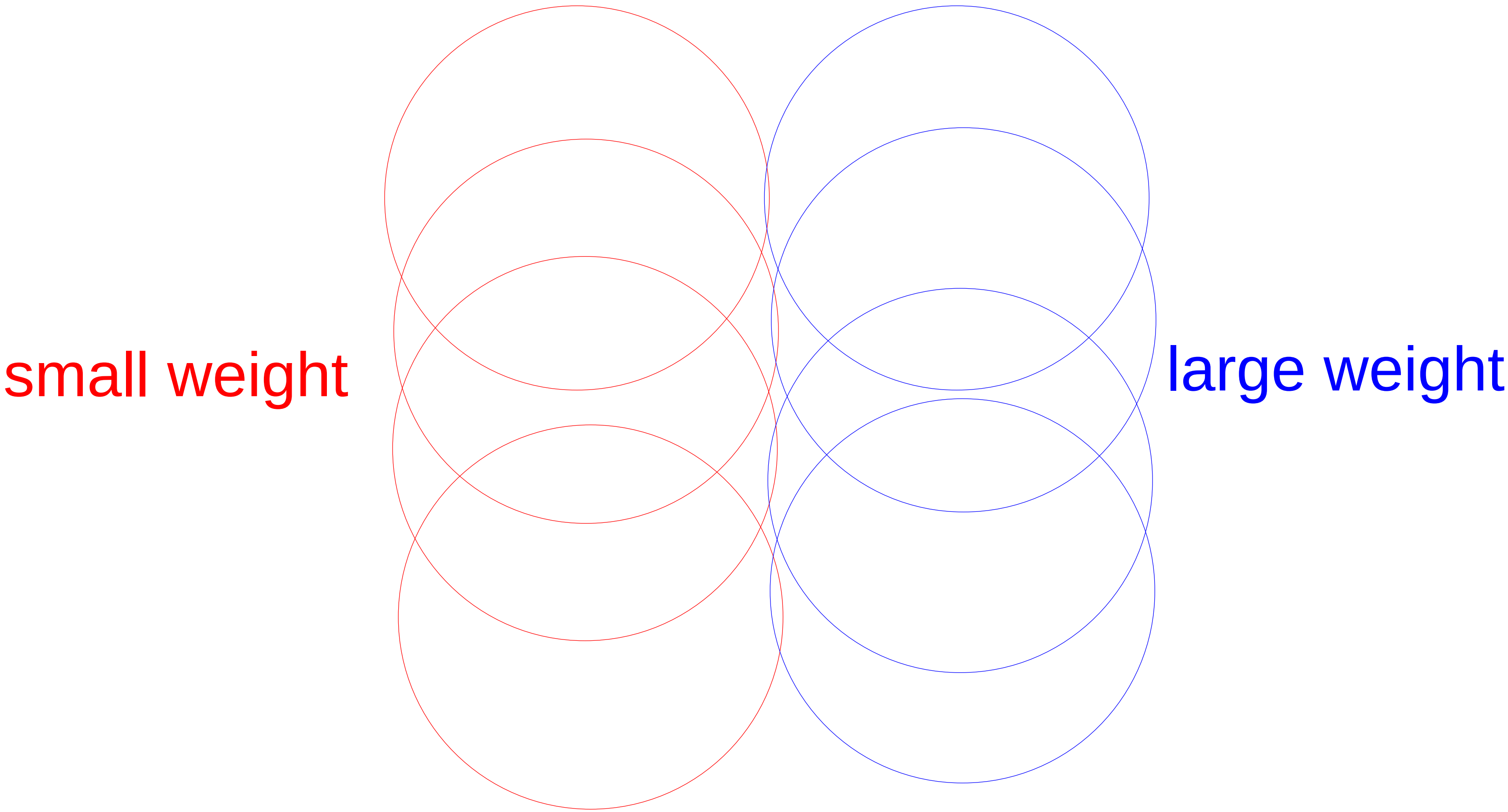
- **Disk graphs:** Shifting + Dynamic Programming
  - Max Weight Ind. Set: PTAS ✔ [EJS01]
  - Min Weight Vertex Cover: PTAS ✔ [EJS01]
  - Min Dominating Set: **???**
  - Min Weight Dominating Set: **???**

Can we use shifting for MWDS in UDG?

MWDS can be arbitrarily large for unit disks in an area of constant size:



➠ Brute-force enumeration no longer works.

Main result

Theorem. There is a constant-factor approximation algorithm for MWDS in unit disk graphs.

Ideas:

- Partition the plane into unit squares and solve the problem for each square separately.
- In each square, reduce the problem to the problem of covering points with weighted disks.
- Use enumeration techniques (guess properties of OPT) and dynamic programming to solve the latter problem.

The constant factor is 72.

The subproblem for each square

- Find a dominating set for the square:
  - Let $\mathcal{D}_S$ denote the set of disks with center in a 1×1 square S.
  - Let $N(\mathcal{D}_S)$ denote the disks in $\mathcal{D}_S$ and their neighbors.
  - **Task:** Find a minimum weight set of disks in $N(\mathcal{D}_S)$ that dominates all disks in $\mathcal{D}_S$.

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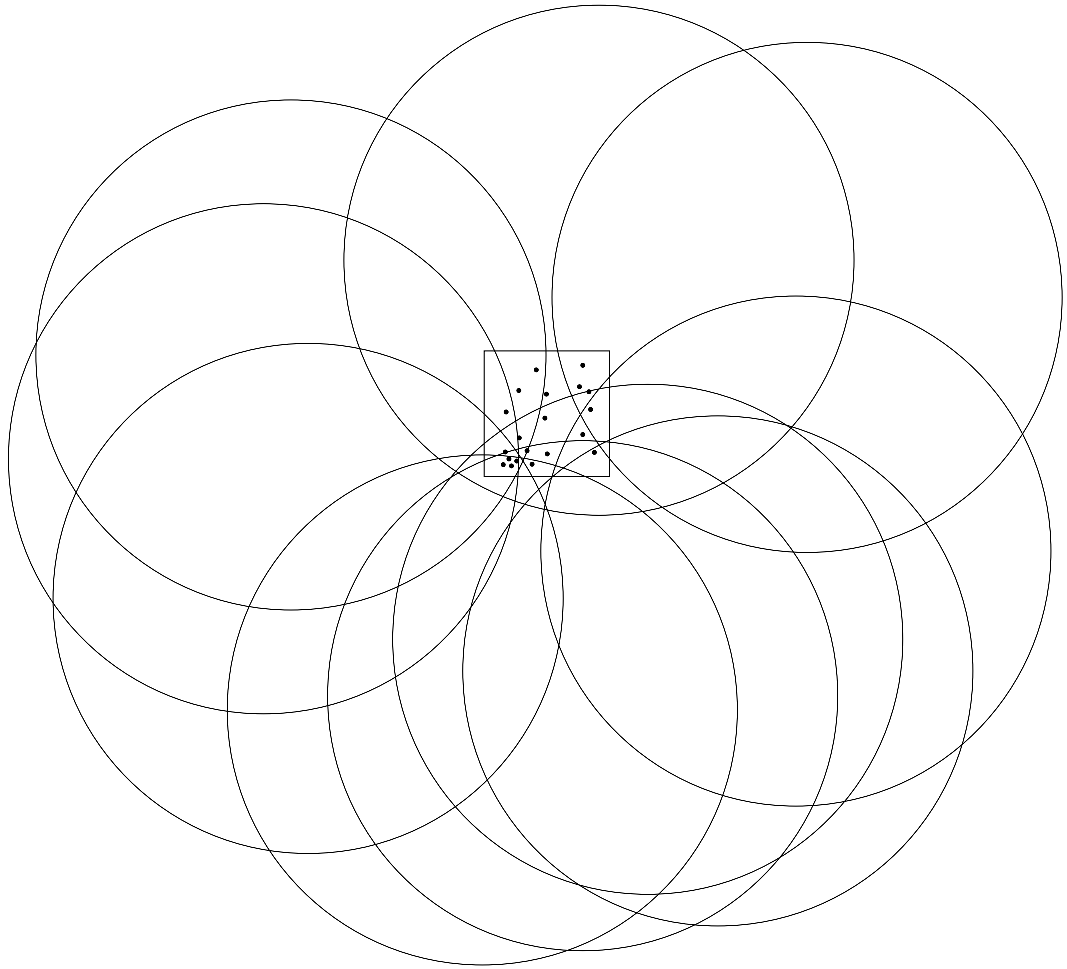
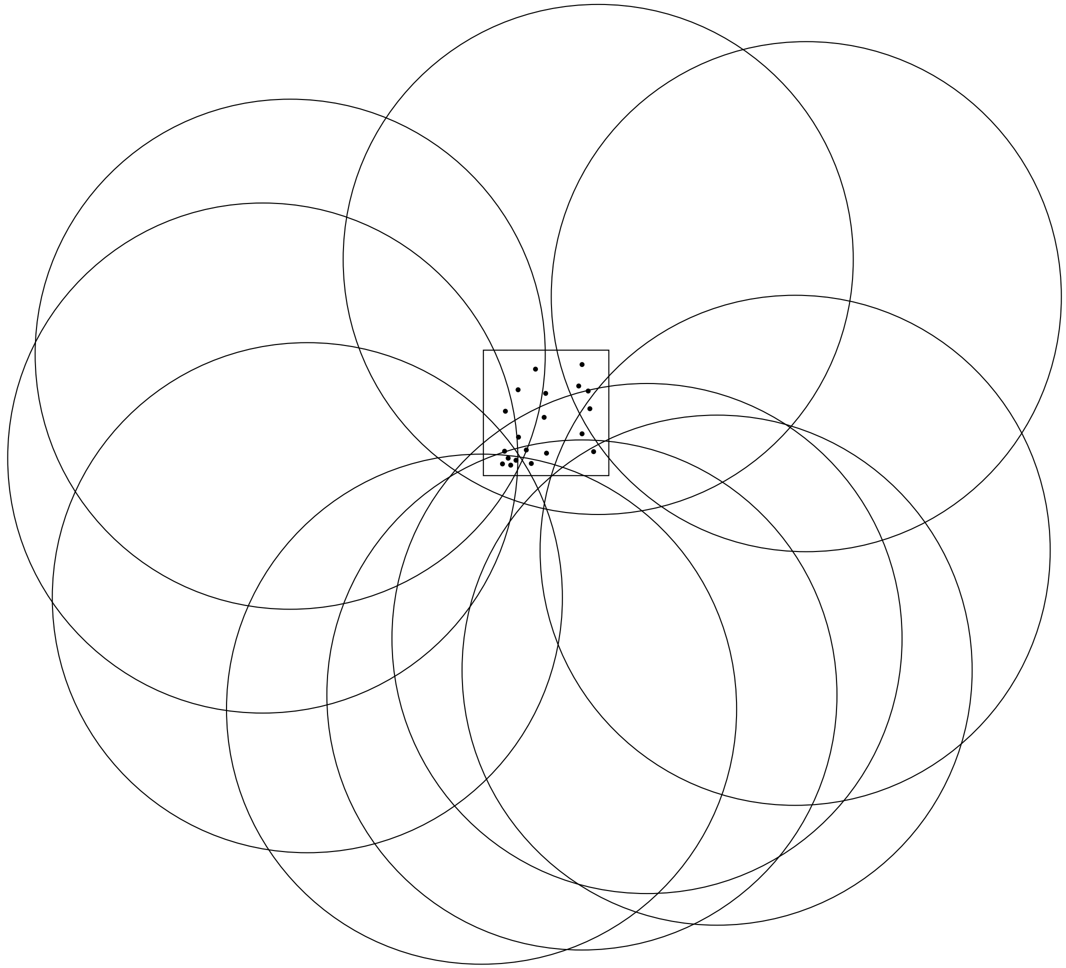
  - **Task:** Find a minimum weight set of disks in $N(\mathcal{D}_S)$ that dominates all disks in $\mathcal{D}_S$.
- Reduces (by guessing the max weight of a disk in OPT$_S$) to covering points in a square with weighted disks:
  - Let P be a set of points in a $\frac{1}{2} \times \frac{1}{2}$ square S.
  - Let $\mathcal{D}$ be a set of weighted unit disks covering P.
  - **Task:** Find a minimum weight set of disks in $\mathcal{D}$ that covers all points in P.

Covering points by weighted disks



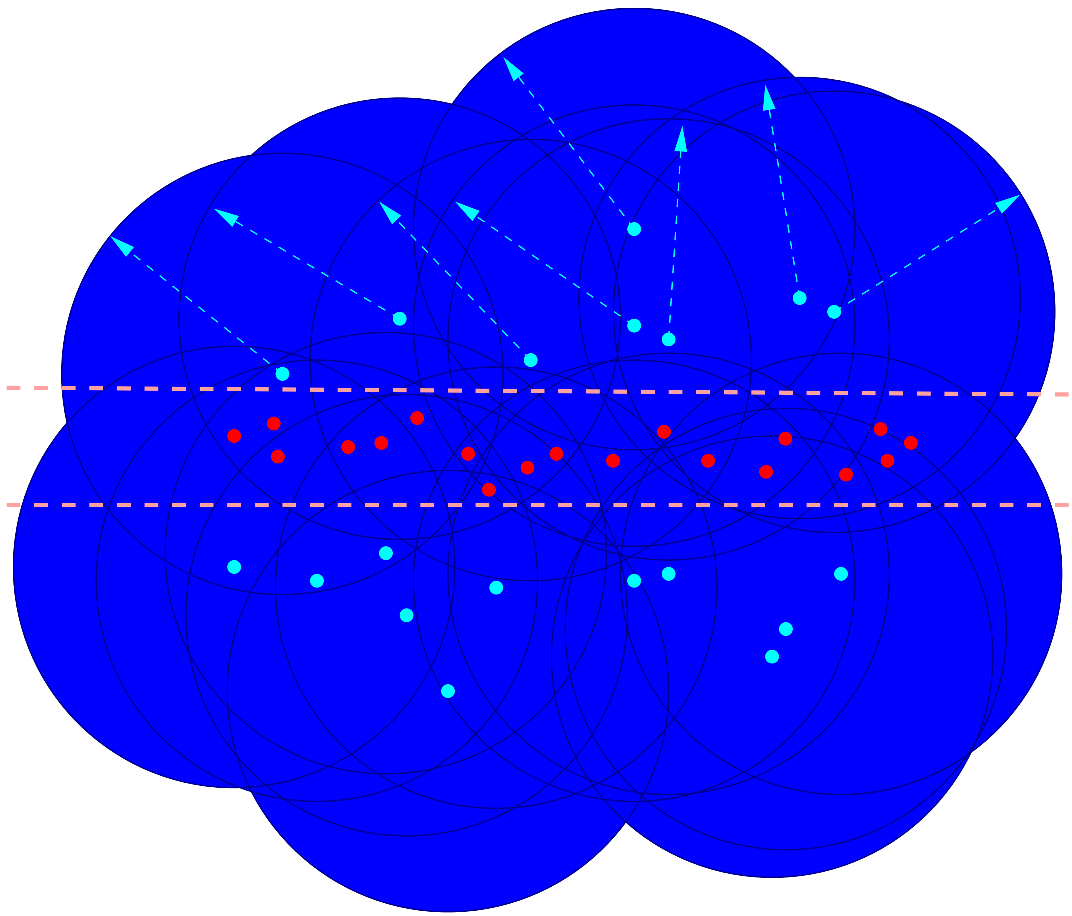
Covering points by weighted disks



Remark. $O(1)$-approximation algorithms are known for unweighted disk cover [Brönninmann and Goodrich, 1995].

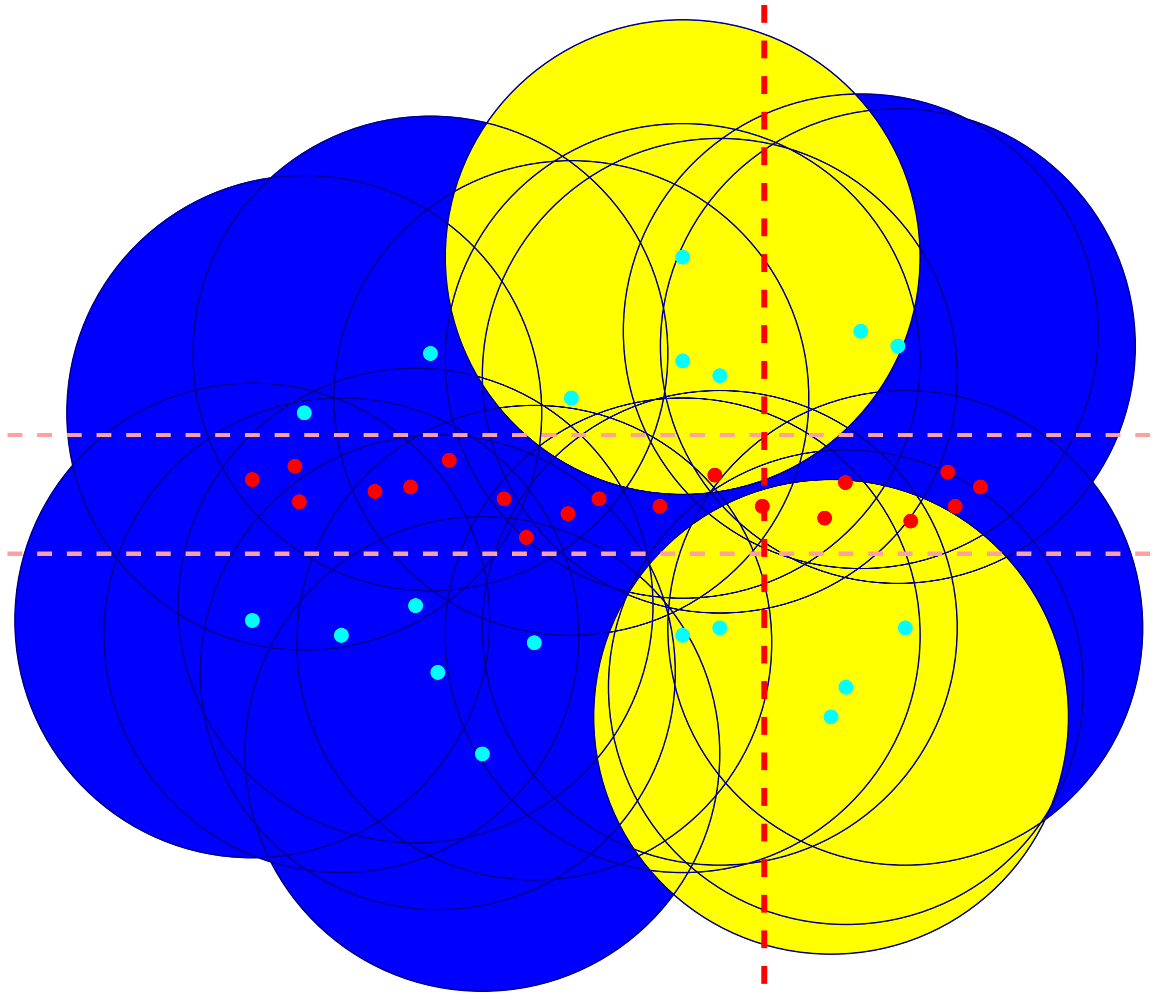
Polynomial-time solvable subproblem

- Given a set of points in a strip, and a set of weighted unit disks with centers outside the strip, compute a minimum weight set of disks covering the points.



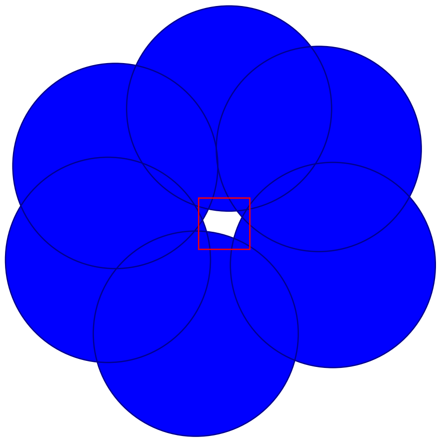
Dynamic programming

- Vertical sweepline, table entry for every pair of disks that could be on the lower and upper envelope:

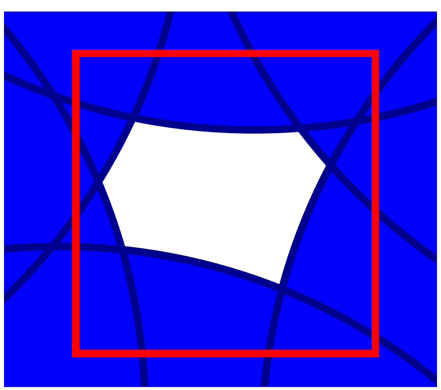


Main cases: One hole or many holes

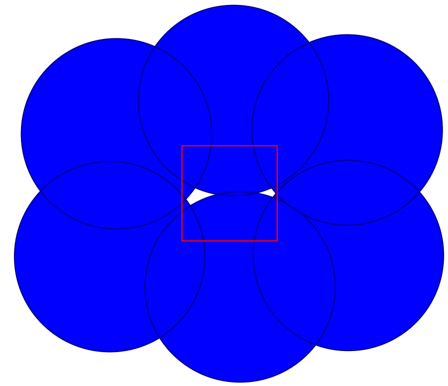
One-hole case:



Enlarged:



Many-holes case:



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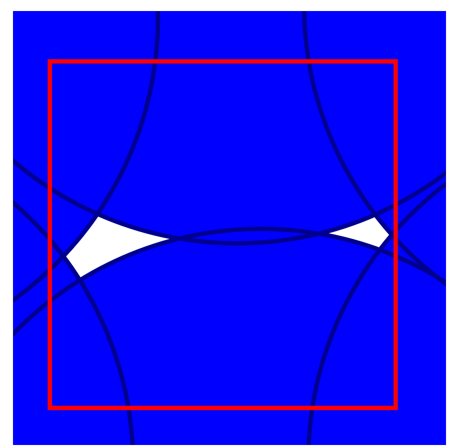
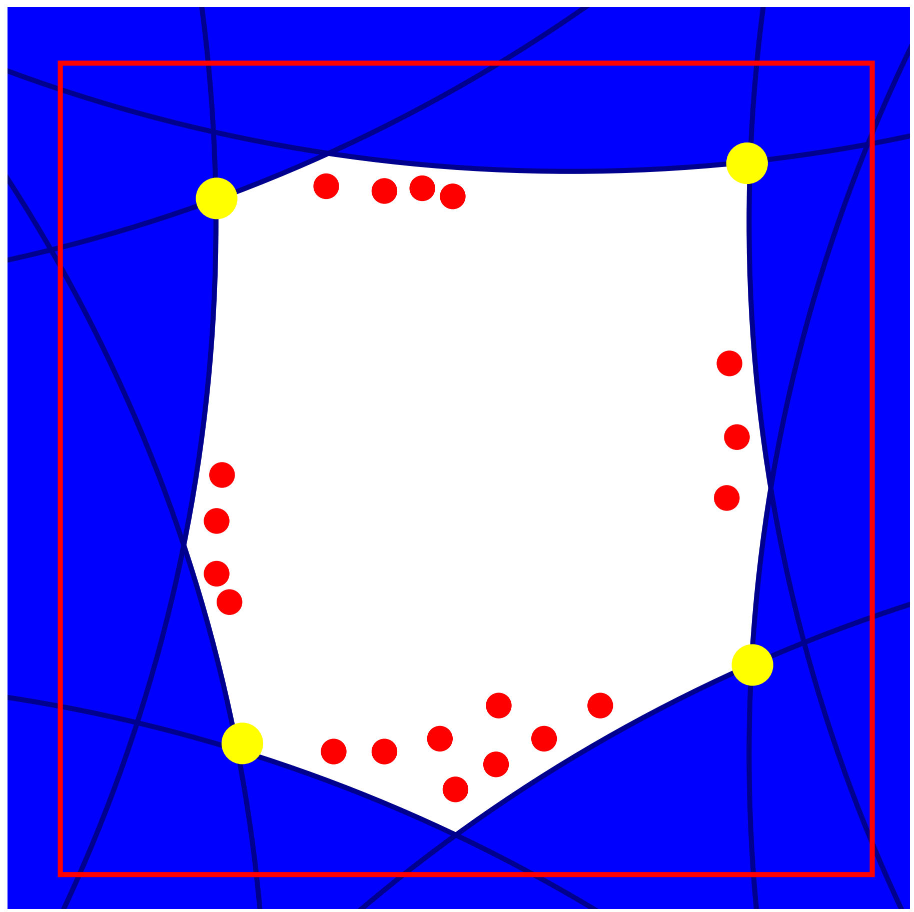


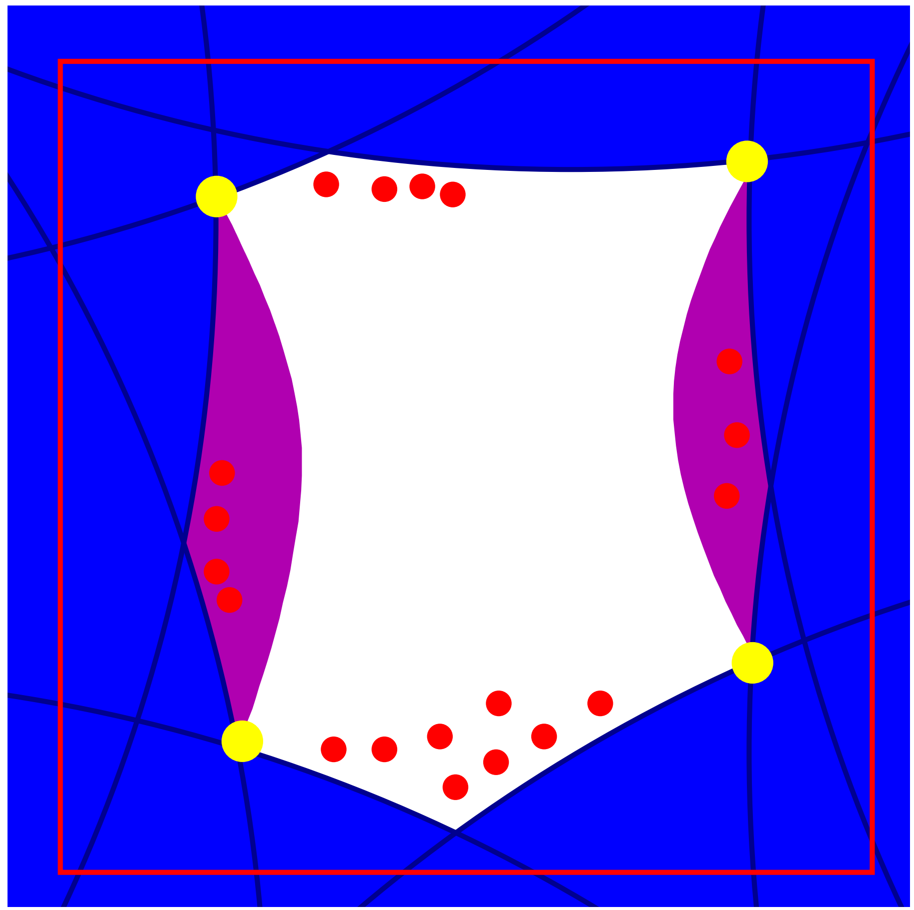
Sketch of the one-hole case

Step 1: Guess the four “corner points” of the optimal solution (each of them is defined by two disks).



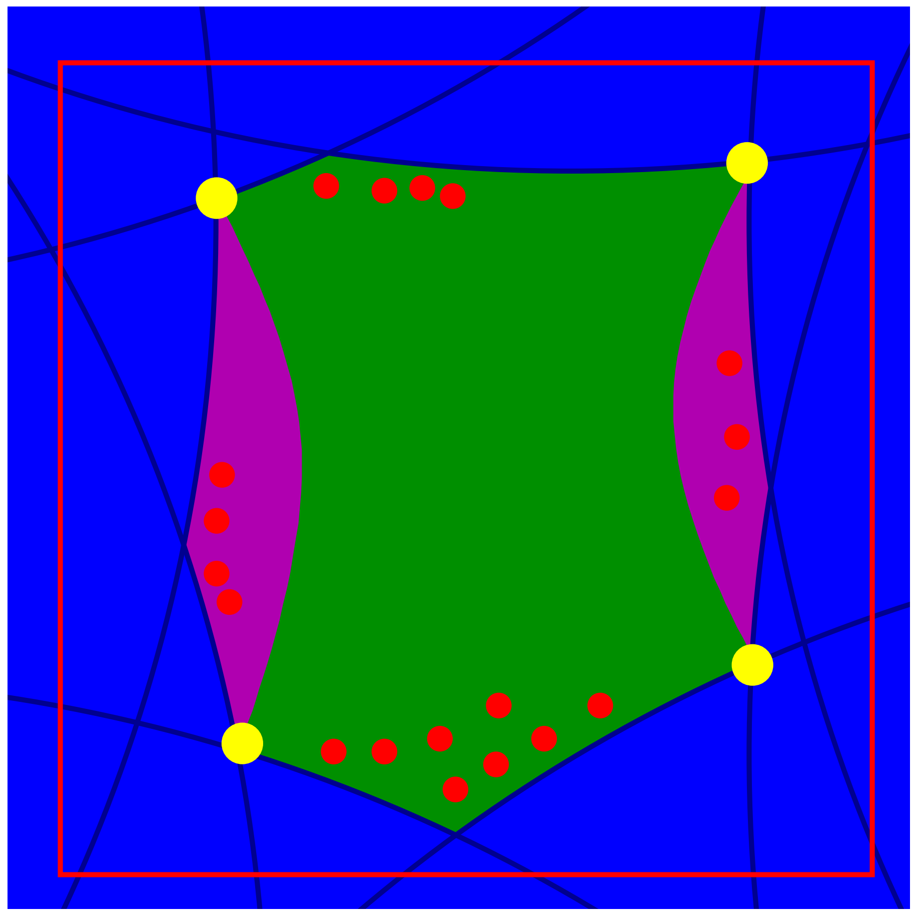
Sketch of the one-hole case

Step 2: Two regions that can only be covered with disks whose centers are to the left or right of the square.



Sketch of the one-hole case

Step 3: Remaining area can only be covered with disks whose centers are above or below the square.



Summary: MWDS in unit disk graphs

- Partition the plane into unit squares and solve the problem for each square separately. (We lose a factor of 36 compared to OPT.)
- For each square, reduce the weighted dominating set problem to a weighted disk cover problem.
- Distinguish one-hole case and many-holes case.
- In each case, we have a 2-approximation or optimal algorithm for covering points in the square with weighted unit disks.
- This implies the constant-factor approximation algorithm for MWDS in unit disk graphs. The ratio is $36 \cdot 2 = 72$.

Weighted Connected Dominating Sets

Theorem. There is a constant-factor approximation algorithm for MWCDS in unit disk graphs.

Algorithm Sketch:

- First, compute an $O(1)$-approximate MWDS D.
- Build auxiliary graph H with a vertex for each component of D, and weighted edges corresponding to paths with at most two internal vertices.
- Compute a minimum spanning tree of H and add the disks corresponding to its edges to D.

We can show: The total weight of the disks added to D is at most $17 \cdot \mathrm{OPT}$, where OPT is the weight of a minimum weight connected dominating set.

Conclusion

- We have presented the first constant-factor approximation algorithms for MWDS and MWCDS in unit disk graphs.
- The approximation ratios are 72 and 89, respectively.
- The MWDS algorithm guesses many properties of the optimal solution and thus has a rather large running-time (exponent > 10)
- **Open Problems**
  - Improve approximation ratio and/or running-time.
  - What approximation ratio can be achieved for weighted or unweighted dominating sets in arbitrary disk graphs?

Thank you!