On the Impact of Combinatorial Structure on Congestion Games

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Congestion Games - Def

Congestion game is a tuple $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$ with

- $\mathcal{N} = \{1, \ldots, n\}$, set of players
- $\mathcal{R} = \{1, \ldots, m\}$, set of resources
- $\Sigma_i \subseteq 2^{[m]}$, strategy space of player i
- $d_r : \{1, \ldots, n\} \rightarrow \mathbb{R}$, delay function or resource r

For any state $S = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \Sigma_n$

- n_r = number of players with $r \in S_i$
- $d_r(n_r)$ = delay of resource r
- $\delta_i(S) = \sum_{r \in S_i} d_r(n_r)$ = delay of player i

S is Nash equilibrium if no player can unilaterally decrease its delay.

Example: Network (Path) Congestion Games

- Given a directed graph $G = (V, E)$ with delay functions $d_e : \{1, \ldots, n\} \to \mathbb{N}$, $e \in E$.
- Player i wants to allocate a path of minimal delay between a source s_i and a target t_i.

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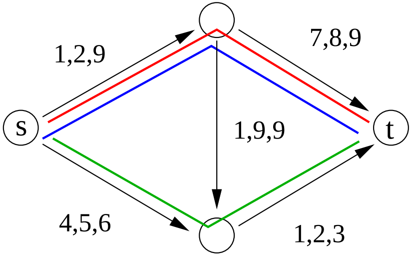

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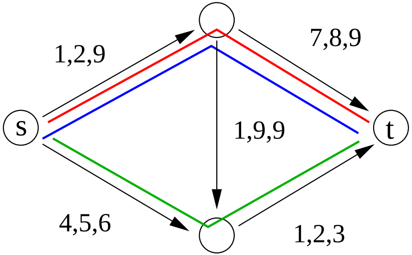

- Game is called *symmetric* if all players have the same source/target pair.

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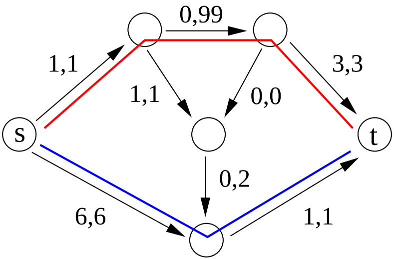
A sequence of (best reply) improvement steps: First step ...



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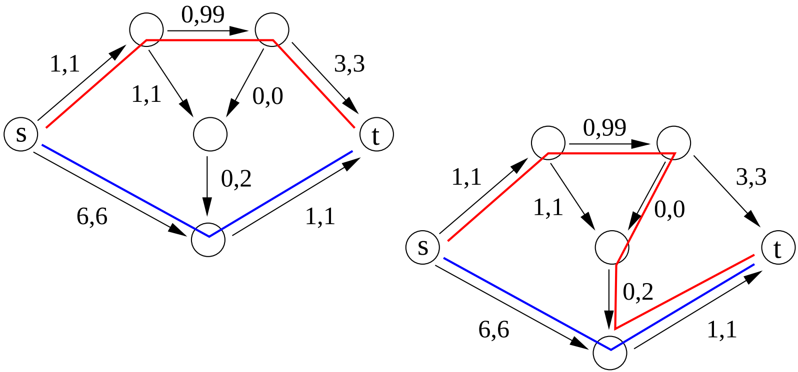

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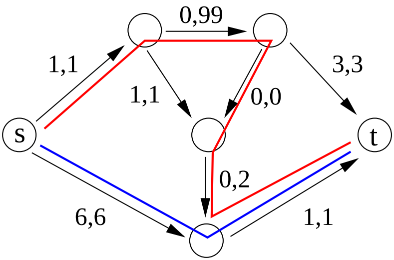

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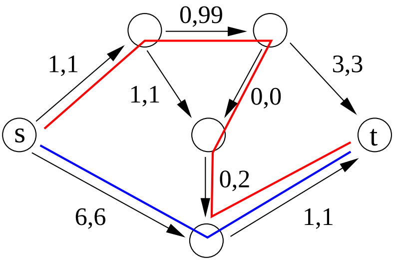

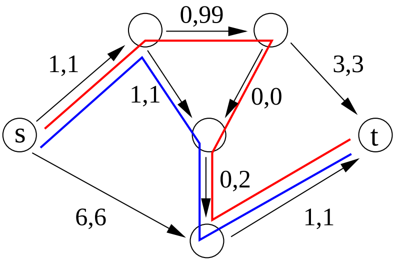

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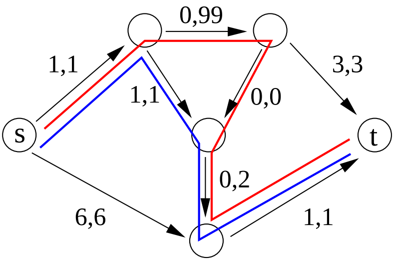

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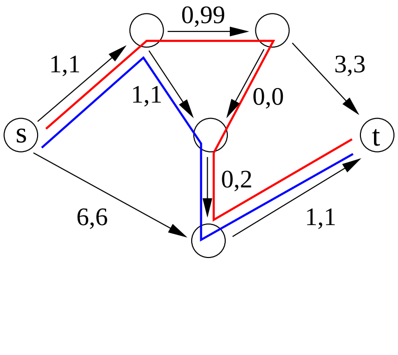

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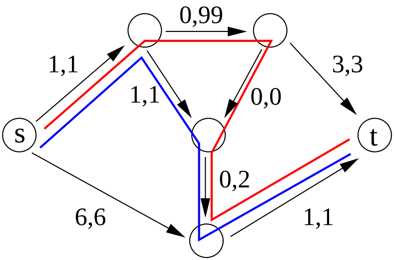

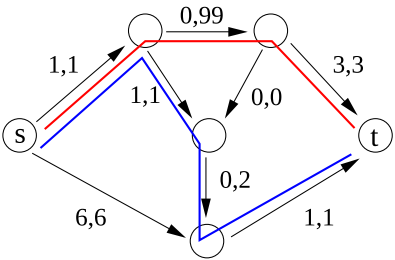

Nash equilibrium – **stop!**

The transition graph

Definition

- The *transisition graph* of a congestion game Γ contains a node for every state S and a directed edge (S, S') if S' can be reached from S by the improvement step of a single player.
- The *best reply transisiton graph* contains only edges for best reply improvement steps.

The sinks of the (best reply) transition graph corresponds to the Nash equilibria of Γ.

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  We will see ...

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Proposition (Rosenthal 1973)

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If a single player decreases its latency by Δ then also the potential decreases by Δ. □

Fast convergence for singleton congestion games

Theorem (Ieong, McGrew, Nudelman, Shoham, Sun, 2005)

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Matroid Congestion Games

Def: Matroid congestion games

- A game Γ is called *matroid congestion game* if, for every $i \in \mathcal{N}$, Σ_i is the bases of a matroid over $\mathcal{R}$.
- All strategies of a player have the same cardinaility, which corresponds to the *rank* of the player's matroid.
- The *rank of the game*, $\mathrm{rk}(\Gamma)$, is defined to be the maximum matroid rank over all players.

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Theorem (Ackermann, Röglin, V., 2006)

In a matroid game Γ, all best response improvement sequences have length $O(n^2\, m\, rk(\Gamma))$.

Matroid Congestion Games: Proof of Fast Convergence

- Sort delay values $d_r(i)$, for $r \in \mathcal{R}$ and $1 \leq k \leq n$, in non-decreasing order.

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Consequence: Rosenthal's potential function yields an upper bound of $n^2\, m\, \mathrm{rk}(\Gamma)$ on the length of a best response sequence as

$$\bar{\phi}(S) = \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} \bar{d}_r(k) \leq \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} n\, m \leq n^2\, m\, \mathrm{rk}(\Gamma) . \qquad \square$$

Fast Convergence beyond the Matroid Property?

Theorem (Ackermann, Röglin, V., 2006)

Let $\mathcal{S}$ be any inclusion-free non-matroid set system. Then, for every n, there exists a 4n-player congestion game with the following properties:

- *the strategy space of each player is isomorph to $\mathcal{S}$,*
- *the delay functions are non-negative and non-decreasing, and*
- *there is a best response sequence of length 2^n.*

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Corollary

The matroid property is the maximal property *on the individual players' strategy spaces that guarantees polynomial convergence.*

Proof Idea for Exponential Convergence

Because of the non-matroid property of instance $\mathcal{I}$, one can show:

1-2-exchange property

There exists three resources a, b, and c with the property that, if the weights of the other resources are set appropriately, an optimal solution of $\mathcal{I}$ contains

- a but not b and c if $w_a < w_b + w_c$, and
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Using this property one can interweave the strategy spaces in form of a counter that yields a best response sequence of length 2^n. □

Further results on the length of best response paths

Fabrikant, Papadimitriou, Talwar, 2004

There are instances of network congestion games that have initial states for which all improvement sequences have exponential length.

Proof technique: PLS-reduction (to be explained next ...)

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Dito for symmetric network congestion games, although Nash equilibria can be found in polynomial time.

Proof technique: embedding of asymmetric network games into symmetric network games

The relationship to local search

Rosenthal's potential function allows to interprete congestion games as local search problems:

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How difficult is it to compute local optima?

The complexity class PLS

PLS (Polynomial Local Search)

PLS contains optimization problems with a specified neighborhood relationship Γ. It is required that there is a poly-time algorithm that, given any solution s,

- computes a solution in $\Gamma(s)$ with better objective value, or
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Examples:

- FLIP (circuit evaluation with Flip-neighborhood)
- Max-Sat with Flip-neighborhood
- Max-Cut with Flip-neighborhood
- TSP with 2-Opt-neighborhood
- Congestion games wrt improvement steps

The complexity class PLS

PLS reductions

Given two PLS problems Π_1 and Π_2 find a mapping from the instances of Π_1 to the instances of Π_2 such that

- the mapping can be computed in polynomial time,
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Examples for PLS-complete problem:

- FLIP (via a master reduction)
- Max-Sat and POS-NAE-SAT
- Max-Cut

Complexity of congestion games

Known Results

	network games	general games
symmetric	$\exists$ poly-time Algo	PLS-complete
asymmetric	PLS-complete	PLS-complete

[Fabrikant, Papadimitriou, Talwar 2004]

Complexity of congestion games

New Results

It is PLS-complete to compute Nash equilibria for the following classes of congestion games:

- threshold congestion games with linear latency functions
- network congestion games with linear latency functions
- undirected network congestion games with linear latency functions
- congestion games for overlay network design with linear latency functions
- market sharing games

Complexity of threshold games

Threshold congestion games:

$\mathcal{R} = \mathcal{R}_{in} \dot{\cup} \mathcal{R}_{out}$. Every player i has two strategies

in: an arbitrary subset $S_i \subseteq \mathcal{R}_{in}$

out: a subset $S'_i = \{r_i\}$ for a unique resource $r_i \in \mathcal{R}_{out}$ with a fixed delay, the so-called *threshold* t_i.

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Theorem (Ackermann, Röglin, V., 2006)

Quadratic threshold games are PLS-complete.

Reduction from Max-Cut to quadratic threshold games

Max-Cut as "party affiliation game"

Nodes correpond to players. The strategies of a node are

- left: choose the left hand side of the cut
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The costs for these strategies are

- left: sum of the weights of the incident edges to the left
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Alternative definition of the costs

- left: sum of the weights of the incident edges to the left
- right: half of the weight of all indident edges

Alternative costs do not change the preferences!

Reduction from Max-Cut to quadratic threshold games

Max-Cut as a quadratic threshhold game

- Nodes correpond to players, edges to resources in $\mathcal{R}_{in}$.
- The strategies of a node are either
  - left: allocate all incident edges from $\mathcal{R}_{in}$
  - right: allocate unique resource from $\mathcal{R}_{out}$
- An edge $e \in \mathcal{R}_{in}$ has the delay function

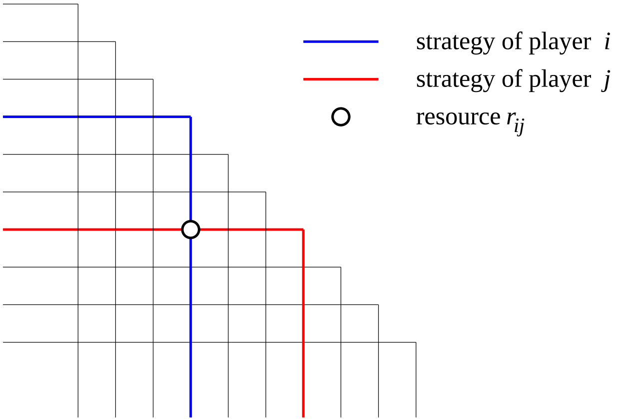
$$d_e(1) = 0 \quad \text{and} \quad d_e(2) = w_e \ .$$

- The threshold for a node v is set to

$$t_v \ = \ \frac{1}{2} \sum_{e \ni v} w_e \ .$$

□

Quadratic threshold games as grid routing games



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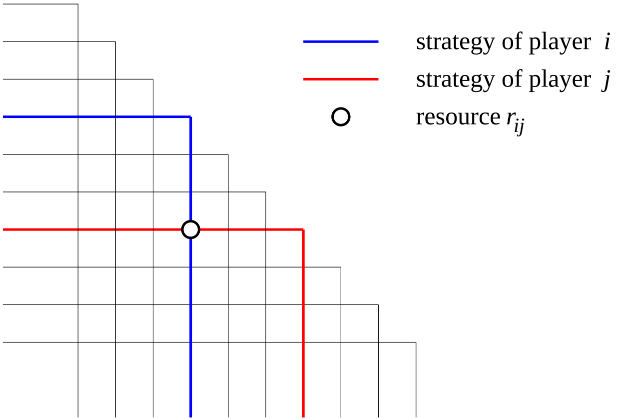

This is the key argument for PLS completeness of network congestion games with linear latency functions. □

Further consequences

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- there are games from all of these classes for which there exist an initial state from which all better response sequences have exponential length, and
- it is PSPACE-hard to compute a Nash equilibrium reachable from a given state.

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