Spectra of random power law graphs *

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*This talk is based on two papers coauthored with Fan Chung and Van Vu. 1. The spectra of random graphs with given expected degrees, *Proceedings of National Academy of Sciences*, **100**, no. 11, (2003), 6313-6318. 2. Eigenvalues of random power law graphs, *Annals of Combinatorics* **7**, (2003), 21–33.

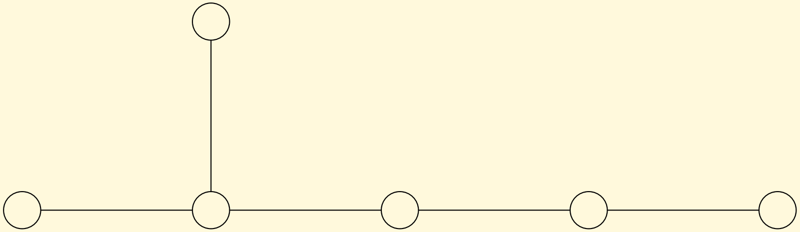
Outline

- Motivation
- The spectrum problem
- Random graphs
- Eigenvalues
- Laplacian spectrum
- Summary

A graph consists of two sets V and E.

- V is the set of vertices (or nodes).

- E is the set of edges, where each edge is a pair of vertices.



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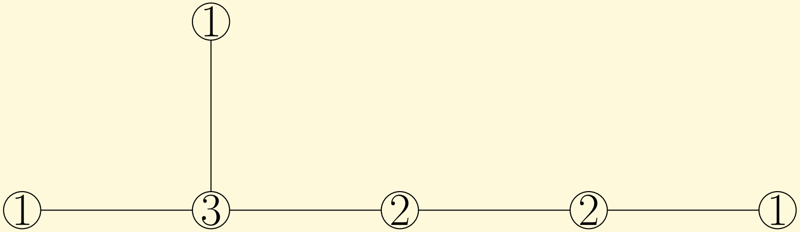


The degree of a vertex is the number of edges, which are incident to that vertex.

1. Motivation

1.1. All these large real-world graphs share a common pattern in their degree distributions.

WWW Graphs

Call Graphs

Collaboration Graphs

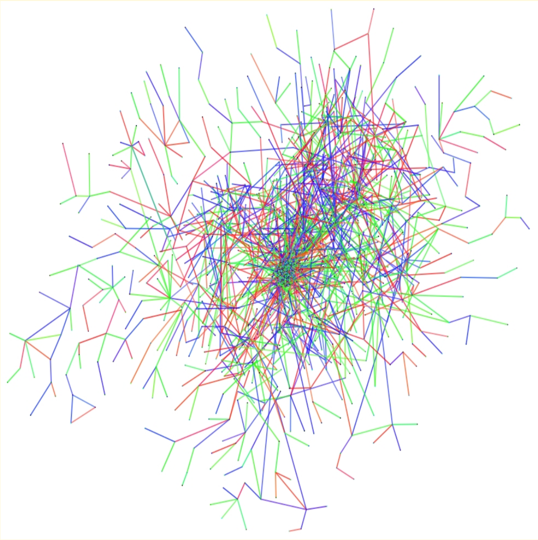
Gene Regulatory Graphs

Frequency of English Words

Graph of U.S. Power Grid

Costars Graph of Actors

⋮



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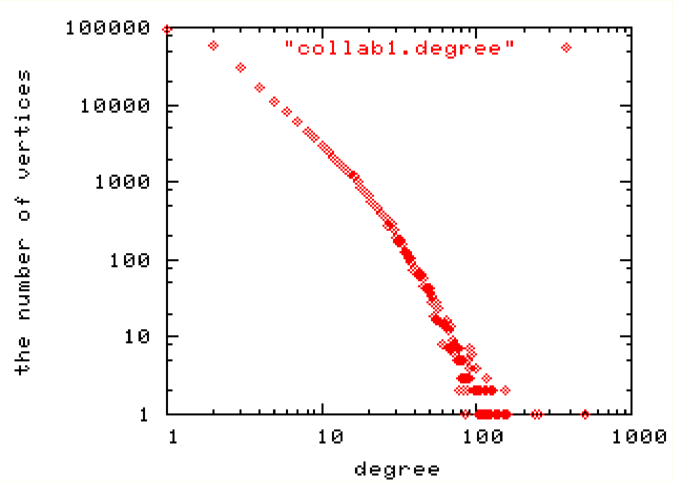
1.2. The power law

The number of vertices of degree k is approximately proportional to $k^{-\beta}$ for some positive β.



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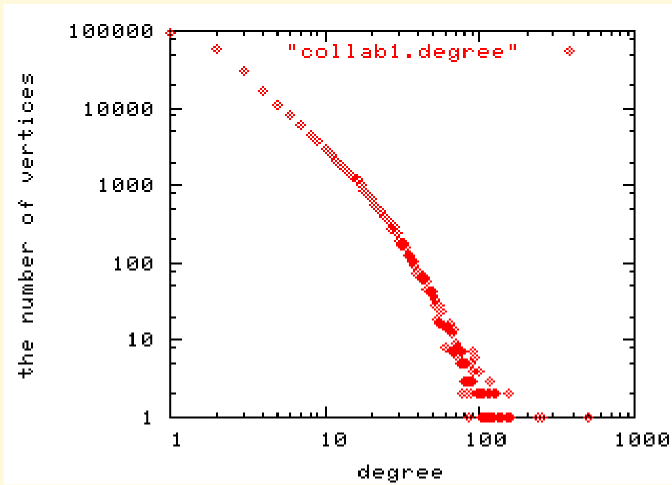

A power law graph is a graph whose degree sequence satisfies the power law.

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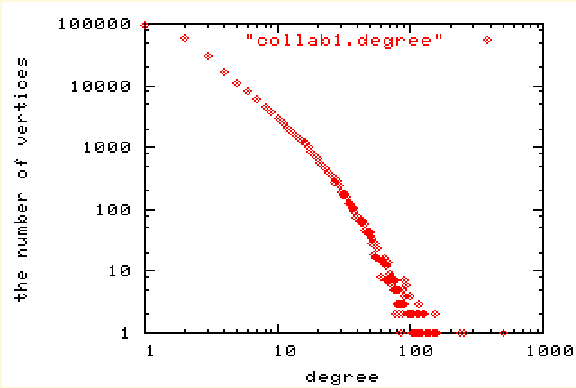

Left: The collaboration graph follows the power law degree distribution with exponent $\beta \approx 3.0$

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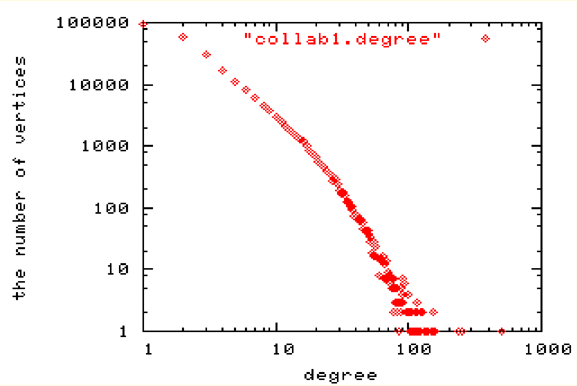
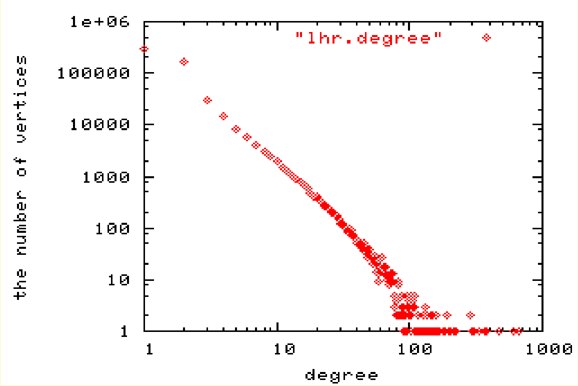
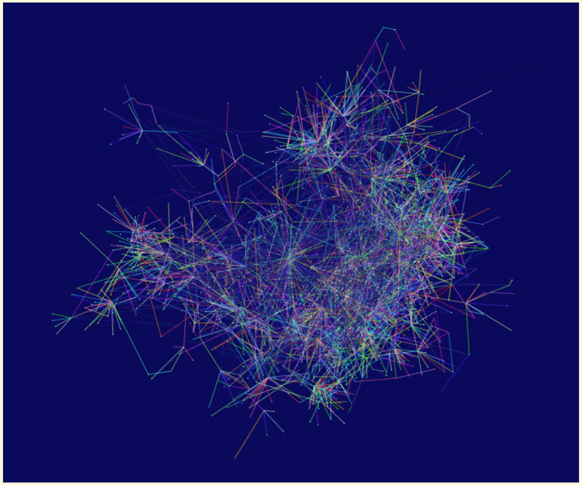


Left: The collaboration graph follows the power law degree distribution with exponent $\beta \approx 3.0$

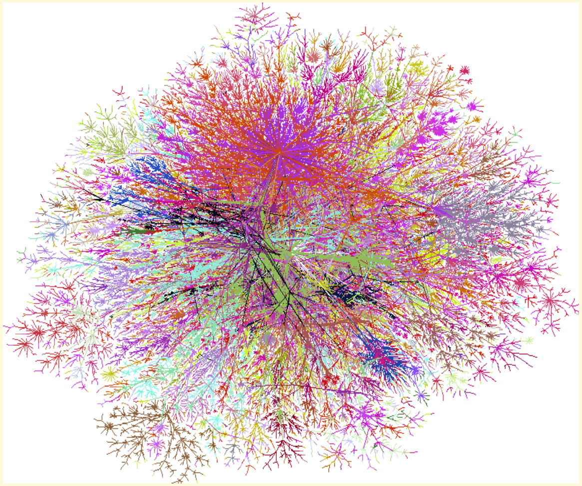
Right: An IP graph follows the power law degree distribution with exponent $\beta \approx 2.4$



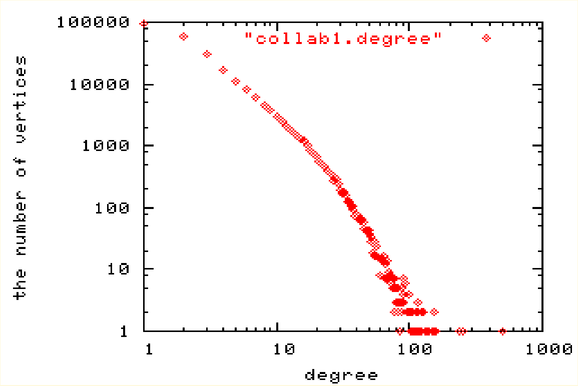
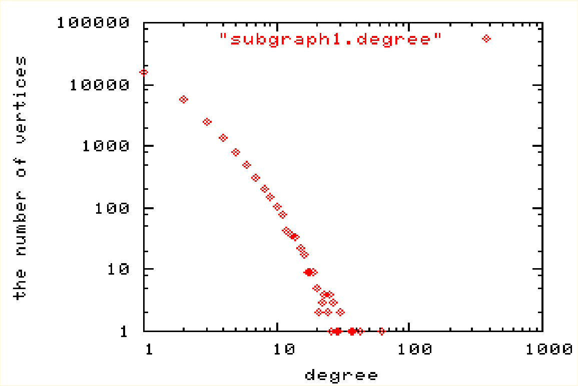


Left: Part of the collaboration graph (authors with Erdős number 2)

Right: An IP graph (by Bill Cheswick)



1.3. The power law degree distribution is robust.

size	degree distribution
25,3339	
52,186	

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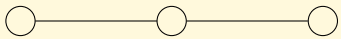
1.4. Eigenvalues of a graph

A graph G:



Adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Eigenvalues are

$$2, 0, 0.$$

2. The spectrum problem

Do the eigenvalues of a power law graph follow the semicircle law or do the eigenvalues have a power law distribution?

2.1. Wigner's semicircle law

(Wigner, 1958)

- A is a real symmetric $n \times n$ matrix.

- Entries a_{ij} are independent random variables.

- $E(a_{ij}^{2k+1}) = 0$.

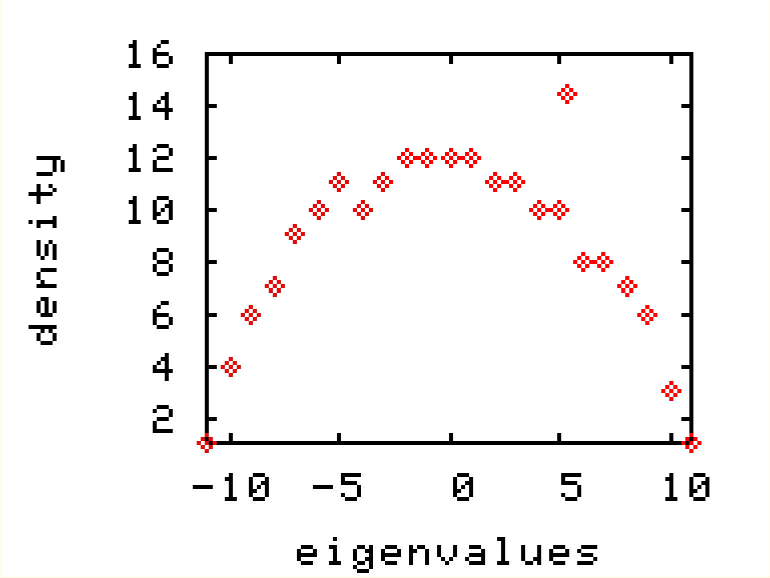
- $E(a_{ij}^2) = m^2$.

- $E(a_{ij}^{2k}) < M$.

The distribution of eigenvalues of A converges into a semicircle distribution of radius $2m\sqrt{n}$.

2.2. Evidence for the semicircle law for power law graphs

The eigenvalues of an Erdős-Rényi random graph follow the semicircle law. (Füredi and Komlós, 1981)



2.3. Evidence against the semicircle law for power law graphs

- **Faloutsos et al. (1999, experimental result)** The eigenvalues of the Internet graph do not follow the semicircle law.
- **Farkas et. al. (2001), Goh et. al. (2001)** The spectrum of a power law graph follows a “triangular-like” distribution.
- **Mihail and Papadimitriou (2002)** They showed that the large eigenvalues are determined by the large degrees. Thus, the significant part of the spectrum of a power law graph follows the power law.

$$\mu_i \approx \sqrt{d_i}.$$

Use random graphs to solve the mystery

- Data sets are too large and dynamic for exact analysis.
- Most real-world graphs have a random or statistical nature.

3. Random graphs

A random graph is a set of graphs together with a probability distribution on that set.

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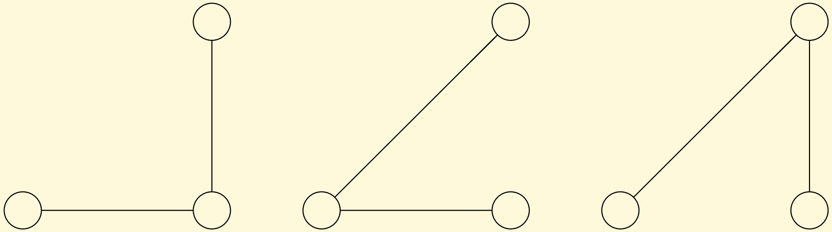
Example: A random graph on 3 vertices and 2 edges with the uniform distribution on it.



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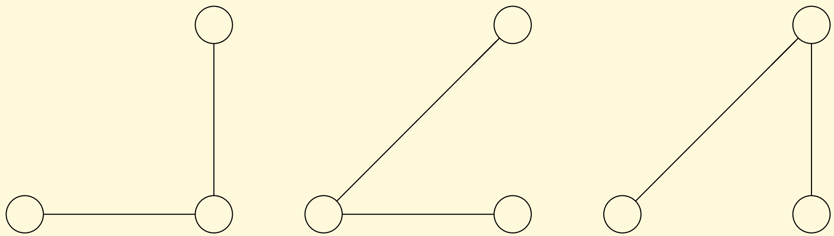

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A random graph G *almost surely* satisfies a property P, if

$$Pr(G \text{ satisfies } P) = 1 - o_n(1).$$

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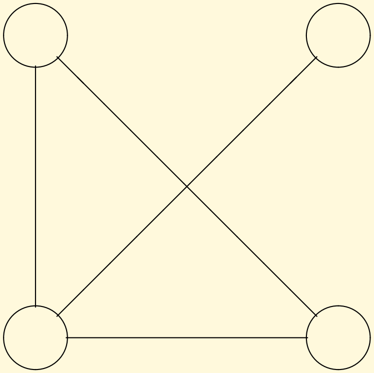

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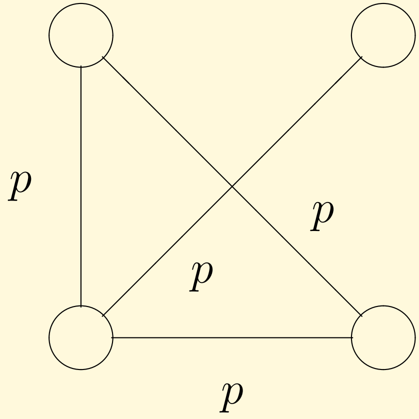

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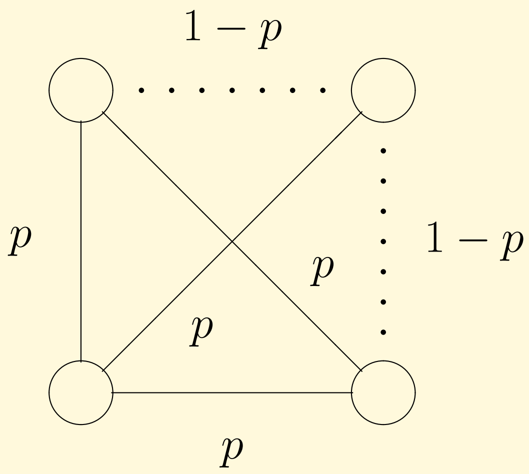

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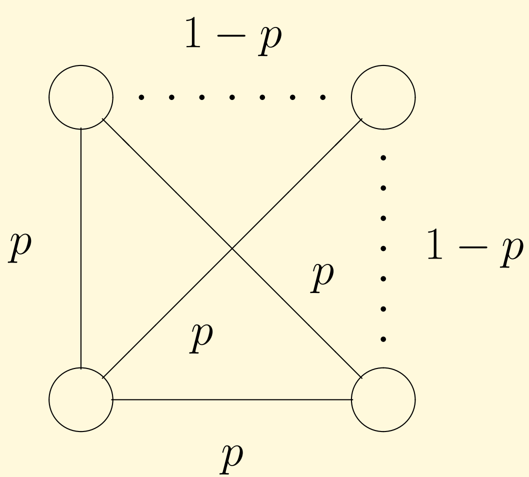

The probability of this graph is

$$p^4(1-p)^2.$$

3.2. A random graph $G(w_1, w_2, \ldots, w_n)$

- n nodes with weights $w_1, w_2, \ldots, w_n$.

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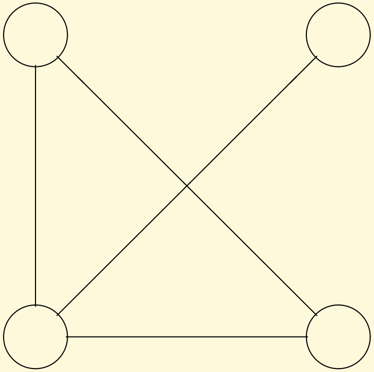

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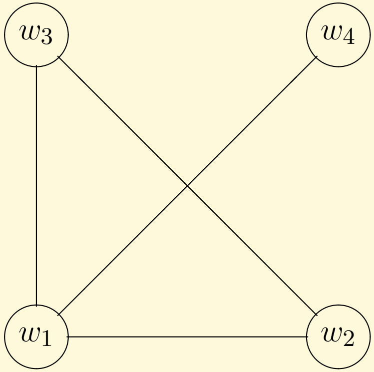

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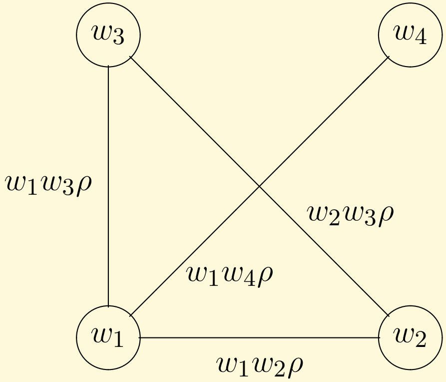

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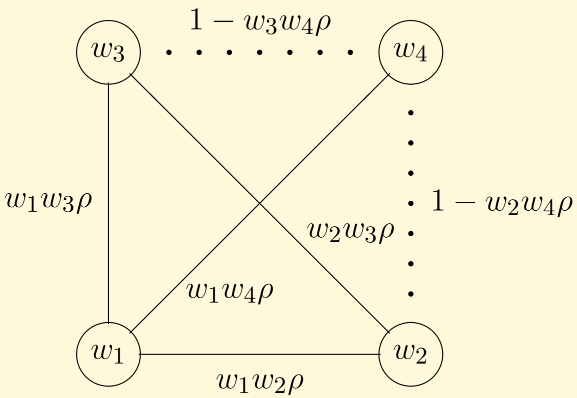

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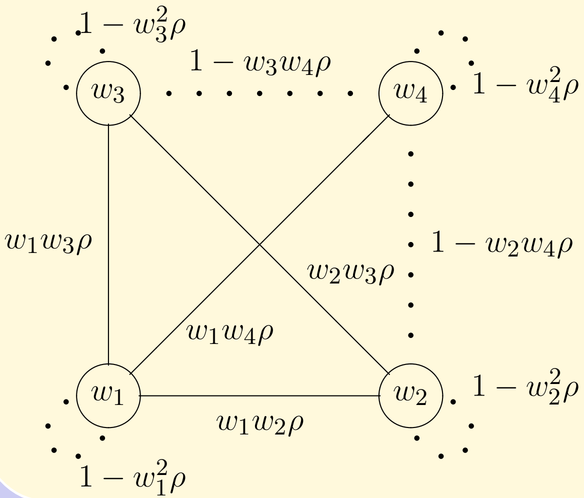

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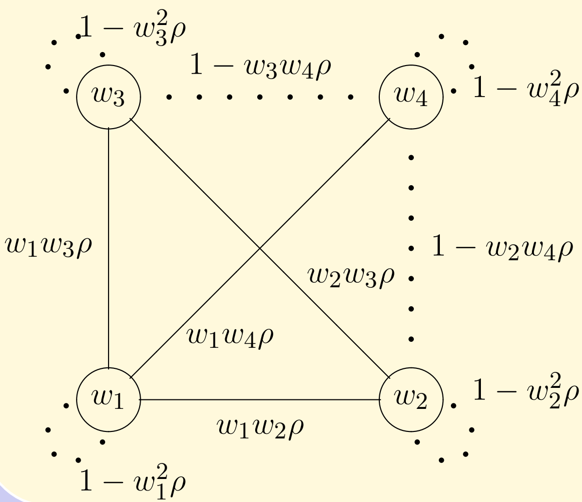

The probability of the graph is

$$w_1^3 w_2^2 w_3^2 w_4 \rho^4 (1 - w_2 w_4 \rho)$$

$$\times (1 - w_3 w_4 \rho) \prod_{i=1}^4 (1 - w_i^2 \rho).$$

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The expected degree of vertex i is

$$E(d_i) = \sum_{j=1}^{n} w_i w_j \rho$$

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Erdős-Rényi model $G(n, p)$ is a special random graph with equal expected degrees:

$$w_1 = \cdots = w_n = np.$$

4. Eigenvalues

Theorem 1 (Chung, Vu, and Lu, 2003)
Suppose $w_1 \geq w_2 \geq \ldots \geq w_n$. Let μ_i be i-th largest eigenvalue of $G(w_1, w_2, \ldots, w_n)$. Let $m = w_1$ and $\tilde{d} = \sum_{i=1}^{n} w_i^2 \rho$. Almost surely we have:

- $(1 - o(1)) \max\{\sqrt{m}, \tilde{d}\} \leq \mu_1 \leq 7\sqrt{\log n} \cdot \max\{\sqrt{m}, \tilde{d}\}$.

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- $\mu_k \approx \sqrt{w_k}$ and $\mu_{n+1-k} \approx -\sqrt{w_k}$, if $\sqrt{w_k} > \tilde{d} \log^2 n$.

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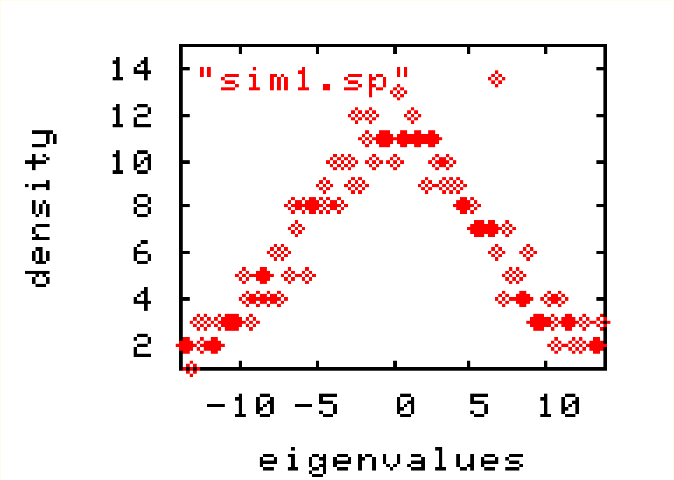
Apply Theorem 1 to the random power law graph:

The first k and last k eigenvalues of the random power law graph with $\beta > 2.5$ follows the power law distribution with exponent $2\beta - 1$. It results a “triangular-like” shape.



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- $\mu_1 \geq X$.
- X can be written as a sum of independent random variables. $X = \frac{1}{\sum_{i=1}^n w_i^2} \sum_{i,j} w_i w_j X_{i,j}$, where X_{ij} is the 0-1 random variable with $Pr(X_{i,j}=1) = w_i w_j \rho$.

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- X can be written as a sum of independent random variables. $X = \frac{1}{\sum_{i=1}^n w_i^2} \sum_{i,j} w_i w_j X_{i,j}$, where X_{ij} is the 0-1 random variable with $Pr(X_{i,j} = 1) = w_i w_j \rho$.
- $E(X) = \tilde{d}$.

Now we will prove $\mu_1 \geq (1 + o(1))\tilde{d}$.

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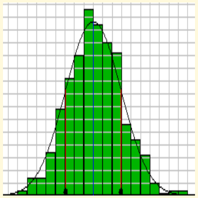
- $E(X) = \tilde{d}$.
- X concentrates on $E(X)$.

Lemma A:
Let $X_1, \ldots, X_n$ be independent random variables with

$$Pr(X_i = 1) = p_i, \qquad Pr(X_i = 0) = 1 - p_i$$

For $X = \sum_{i=1}^{n} a_i X_i$, we have $E(X) = \sum_{i=1}^{n} a_i p_i$ and we define $\nu = \sum_{i=1}^{n} a_i^2 p_i$. Then we have

$$Pr(X < E(X) - t) \ \leq \ e^{-\frac{t^2}{2\nu}};$$
$$Pr(X > E(X) + t) \ \leq \ e^{-\frac{t^2}{2(Var(X)+at/3)}};$$



where a the maximum coefficient among a_i's.

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Lemma B:

$$\mu_1 \leq \tilde{d} + \sqrt{6\sqrt{m \log n}(\tilde{d} + \log n)} + 3\sqrt{m \log n}.$$

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Proof of Lemma B: For a fixed value x (to be chosen later), we define $C = diag(c_1, c_2, \ldots, c_n)$ as follows:

$$c_i = \begin{cases} w_i & \text{if } w_i > x \\ x & \text{otherwise.} \end{cases}$$

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$$E(X_i) \le \tilde{d} + x;$$

$$Var(X_i) \le \frac{m}{x}\tilde{d} + x.$$

By Lemma A, we have

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The outline for proving $\mu_k = (1 + o(1))\sqrt{w_k}$.

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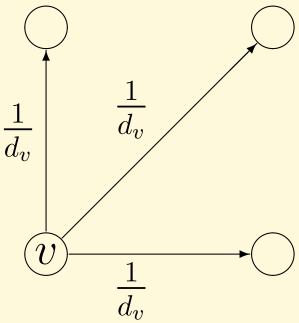
- $G(S,T)$ contains a subgraph G_1 which is a disjoint union of stars with sizes $(1+o(1))w_1, \ldots, (1+o(1))w_k$.
- The maximum degrees m_S and m_T of $G_2 = G(S,T) \setminus G_1$ are small. We have

$$\mu_1(G_2) \le \sqrt{m_S m_T} = o(\sqrt{w_k}).$$

5. Laplacian spectrum

Random walks on a graph G:

$$\pi_{k+1} = AD^{-1}\pi_k.$$
$$AD^{-1} \sim D^{-1/2}AD^{-1/2}.$$



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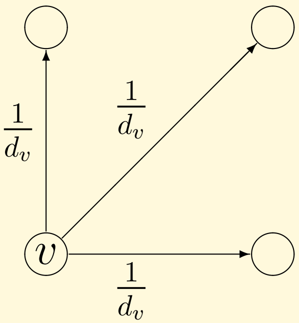

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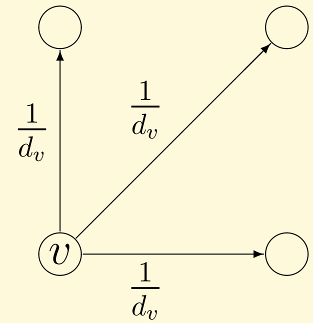

Laplacian spectrum

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The eigenvalues of AD^{-1} are $1, 1-\lambda_1, \dots, 1-\lambda_{n-1}$.

Let

- $w_{min} = \min\{w_1, \ldots, w_n\}$
- $d = \frac{1}{n}\sum_{i=1}^{n} w_i$
- $g(n)$ — a function tending to infinity arbitrarily slowly.

Theorem 2 (Chung, Vu, and Lu, 2003)
If $w_{\min} \gg \log^2 n$, then almost surely the Laplacian spectrum λ_i's of $G(w_1, \ldots, w_n)$ satisfy

$$\max_{i\neq 0} |1-\lambda_i| \leq (1+o(1))\frac{4}{\sqrt{d}} + \frac{g(n)\log^2 n}{w_{\min}}.$$

$$M = D^{-1/2} A D^{-1/2} - \phi_0^* \phi_0$$

where

$$\phi_0 = \frac{1}{\sqrt{\sum_{i=1}^n d_i}} (\sqrt{d_1}, \ldots, \sqrt{d_n})^*.$$

$$C = W^{-1/2} A W^{-1/2} - \chi^* \chi$$

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It is enough to estimate the spectrum of C.

5.1. Results on spectrum of C

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- If $w_{\min} \gg \sqrt{d}$, the eigenvalues of C follow the semi-circle distribution with radius $r \approx \frac{2}{\sqrt{d}}$.

5.2. The proof

Wigner's high moment method:

$$\|C\| \leq \text{Trace}(C^{2k})^{\frac{1}{2k}}.$$

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First we will bound $E(\text{Trace}(C^{2k}))$.

$$E(\text{Trace}(C^{2k})) = \sum_{i_1,i_2,\ldots,i_{2k}} E(c_{i_1 i_2} c_{i_2 i_3} \cdots c_{i_{2k-1} i_{2k}} c_{i_{2k} i_1})$$

$$= \sum_{l \ge 1} \sum_{I_l} \prod_{h=1}^{l} E(c_{e_h}^{m_h})$$

$I_k = \{$ closed walks of length $2k$ which use l different edges $e_1, \ldots, e_l$ with corresponding multiplicities $m_1, \ldots, m_l.\}$

$$E(c_{e_h}) \;=\; 0,$$

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We have

$$E(\text{Trace}(C^{2k})) \leq \sum_{l=1}^{l} W_{l,k} \frac{\rho^l}{w_{min}^{2k-2l}}.$$

Here $W_{l,k}$ denotes the set of closed good walks on K_n of length $2k$ using exactly l different edges.

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$$|W_{l,k}| \le n(n-1)\dots(n-l)\binom{2k}{2l}\binom{2l}{l}\frac{1}{l+1}(l+1)^{4(k-l)}.$$

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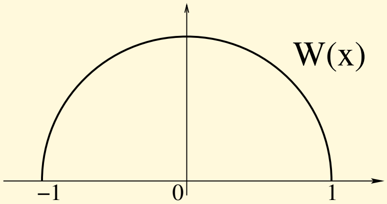
if $k \gg \log n$.

5.3. The proof of the semicircle law

Let $W(x)$ be the cumulative distribution function of the unit semicircle.



$$\int_{-1}^{1} x^{2k} dW(x) = \frac{(2k)!}{2^{2k} k!(k+1)!}$$

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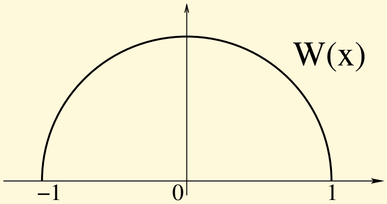

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Let $C_{\text{nor}} = (\frac{2}{\sqrt{d}})^{-1} C$. Let $N(x)$ be the number of eigenvalues of C_{nor} less than x and $W_n(x) = n^{-1} N(x)$ be the cumulative distribution function.

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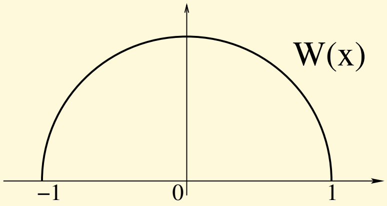

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$$\int_{-\infty}^{\infty} x^{2k} dW_n(x) = \frac{1}{n} E(\text{Trace}(C_{\text{nor}}^{2k})) = \frac{(1+o(1))(2k)!}{2^{2k} k!(k+1)!},$$

$$\int_{-\infty}^{\infty} x^{2k+1} dW_n(x) = \frac{1}{n} E(\text{Trace}(C_{\text{nor}}^{2k+1})) = o(1).$$

Thus, $W_n(x) \longrightarrow W(x)$ (in probability) as $n \to \infty$.

6. Summary

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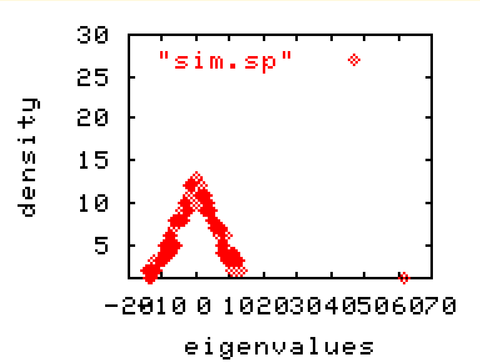

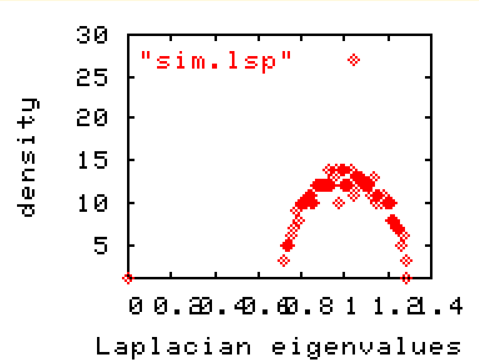

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Applied the above results to random power law graphs, we have

Range	Average distance	Diameter
$2 < \beta < 3$	$\Theta(\log \log n)$	$\Theta(\log n)$
$\beta = 3$	$\Theta(\frac{\log n}{\log \log n})$	$\Theta(\log n)$
$\beta > 3$	$(1+o(1))\frac{\log n}{\log \tilde{d}}$	$\Theta(\log n)$

[a]References: 1.Connected components in a random graph with given degree sequences, *Annals of Combinatorics* **6**, (2002), 125–145. 2. The average distance in random graphs with given expected degrees, *Proceedings of National Academy of Sciences*, **99** (2002), 15879-15882.